MATH 009C **FINAL** **August 11, 2018**

Name: ______________________________ **Score:** _______ / 100

Student ID: _________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										80
Score										
Pts. Possible	10	10	10	10	10	10	10	10	10	85

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours to complete the exam.
- The test will be out of **80** points (8 questions). You may attempt a 9^{th} question, which will have a maximum of 5 possible points. The highest possible score is therefore **85** points.
- In the row with the ✓, mark with a ✓ the problems you want graded for credit, and **EC** for your extra credit problem. **If you do not mark the boxes, problems 1-8 will be graded for credit regardless of which ones you complete, and 9 will be your extra credit.**
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. **Any unjustified claims will receive no credit. This means you need to state which test you are using for series questions!** Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

FORMULAS:

Common Taylor Series	Common Taylor Series
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for all } \lvert x\rvert < 1$	$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all } x \in \mathbb{R}$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x \in \mathbb{R}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for all } x \in \mathbb{R}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad \text{for } x \in (-1, 1]$	$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{for } \lvert x\rvert \leq 1$
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad \text{for } \lvert x-a\rvert < R$	$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n, \quad \text{for } \lvert x\rvert < 1$

1) (5 pts.) (a) Determine whether the sequence converges or diverges:

$$a_n = \frac{\arctan(n)}{e^{-n} - 1}.$$

(5 pts.) (b) Determine whether the sequence converges or diverges:

$$a_n = \frac{(-1)^n + n}{(-1)^n - n}.$$

Solution:

(a) By taking the limit directly, we get

$$\lim_{n\to\infty} \frac{\arctan(n)}{e^{-n} - 1} = \frac{\frac{\pi}{2}}{0 - 1} = -\frac{\pi}{2},$$

so since the limit is a finite number, the sequence a_n converges.

(b) Using the trick of multiplying the numerator and denominator by $\frac{1}{n}$ in the limit, we get

$$\begin{aligned}\lim_{n\to\infty} \frac{(-1)^n + n}{(-1)^n - n} &= \lim_{n\to\infty} \frac{(-1)^n + n}{(-1)^n - n} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} \\ &= \lim_{n\to\infty} \frac{\frac{(-1)^n}{n} + 1}{\frac{(-1)^n}{n} - 1} = \lim_{n\to\infty} \frac{0+1}{0-1} \\ &= -1\end{aligned}$$

where we apply the theorem from class that states that if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. We do the computation

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

so the alternating term goes to zero by the theorem. Since the limit is equal to -1, the sequence converges.

2) (5 pts.) (a) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

$$\sum_{n=1}^{\infty} \arctan(n).$$

(5 pts.) (b) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

$$\sum_{n=1}^{\infty} 2^{1-n}3^{-3n}.$$

Solution:

(a) Use the Divergence Test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} \neq 0.$$

Since the limit is not equal to zero, the series diverges.

(b) The series can be put into geometric series form:

$$\begin{aligned}
\sum_{n=1}^{\infty} 2^{1-n}3^{-3n} &= \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} \cdot \frac{1}{3^{3n}} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{3^3}\right)^{n} \\
&= \sum_{n=1}^{\infty} \frac{1}{27} \left(\frac{1}{2}\right)^{n-1} \left(\frac{1}{27}\right)^{n-1} \\
&= \sum_{n=1}^{\infty} \frac{1}{27} \left(\frac{1}{54}\right)^{n-1} \\
&= \frac{\frac{1}{27}}{1-\frac{1}{54}} = \frac{1}{27} \cdot \frac{54}{53} = \frac{2}{53},
\end{aligned}$$

where $a = \dfrac{1}{27}$ and $r = \dfrac{1}{54}$, so the geometric series is convergent. We then compute the sum using the geometric series formula.

3) (10 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=3}^{\infty} \frac{\ln(n)}{n}.$$

Solution:

Use the Integral Test. Define $f(x) = \dfrac{\ln(x)}{x}$, so that we have $a_n = f(n)$. Now check the conditions for the integral test.

- $f(x)$ is continuous on $[3, \infty)$ as its only undefined at $x = 0$, which is not in $[3, \infty)$.
- $f(x)$ is positive as $x > 0$ and $\ln(x) > 0$ for $x \geq 3$.
- $f'(x) = \dfrac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \dfrac{1 - \ln(x)}{x^2} < 0$ for $1 - \ln(x) < 0$ or $e < x$. Therefore, $f(x)$ is decreasing after $x = e$, so $f(x)$ is decreasing on $[3, \infty)$.

Then by computation

$$\begin{aligned}
\int_3^{\infty} \frac{\ln(x)}{x}\, dx &= \lim_{t\to\infty} \int_3^t \frac{\ln(x)}{x}\, dx \\
&= \lim_{t\to\infty} \int_{\ln(3)}^{\ln(t)} u\, du \qquad u = \ln(x), \quad du = \frac{1}{x}\, dx \\
&= \left.\frac{u^2}{2}\right|_{\ln(3)}^{\ln(t)} \\
&= \frac{1}{2} \lim_{t\to\infty} (\ln(t))^2 - (\ln(3))^2) \\
&= \infty.
\end{aligned}$$

So the series is divergent by Integral Test.

4) (10 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{5n}{n^2 + |\sin(n)|}.$$

Solution:

Use the Limit Comparison Test as in Worksheet 6. The terms in the series are all positive, so we can apply limit comparison test. For these problems, take the highest powers of the numerator and the denominator, including any roots that are present to find out what to compare with

$$\frac{5n}{n^2 + |\sin(n)|} \quad \Rightarrow \quad \frac{n}{n^2} = \frac{1}{n}.$$

So the series $\sum_{n=1}^{\infty} \frac{1}{n}$ would give a divergent p-series. So we expect this series to be divergent. Take $\sum a_n$ to be our series and $\sum b_n = \sum \frac{1}{n}$. We now compute the limit and use some algebra

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5n}{n^2+|\sin(n)|}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{5n^2}{n^2 + |\sin(n)|} \\
&= \lim_{n\to\infty} \frac{5n^2}{n^2 + |\sin(n)|} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{5}{1 + \frac{|\sin(n)|}{n^2}} = \frac{5}{1} = 5
\end{aligned}$$

Formally, we show the term with sine goes to zero by squeeze theorem

$$\begin{aligned}
0 \le \frac{|\sin(n)|}{n^2} &\le \frac{1}{n^2} \\
\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{|\sin(n)|}{n^2} &\le \lim_{n\to\infty} \frac{1}{n^2} \\
0 \le \lim_{n\to\infty} \frac{|\sin(n)|}{n^2} &\le 0 \\
\Rightarrow \quad \lim_{n\to\infty} \frac{|\sin(n)|}{n^2} &= 0
\end{aligned}$$

Since the limit is a finite positive number, our series is divergent since $\sum b_n$ is divergent. So by limit comparison test, the series diverges.

5) (10 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}.$$

<u>Solution</u>:

Use the Ratio Test as we have factorials present. We get

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{((n+1)!)^2}\cdot\frac{(n!)^2}{n^n} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{n^n}\cdot\frac{n!n!}{(n+1)!(n+1)!} \\
&= \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{n^n}\cdot\frac{n!n!}{(n+1)(n+1)n!n!} \\
&= \lim_{n\to\infty} \frac{1}{n+1}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^n}{n+1} \\
&= \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n}{\lim_{n\to\infty} n+1} = \frac{e}{\infty} = 0
\end{aligned}$$

So the series is absolutely convergent by the Ratio Test.

6) (10 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt[6]{n^6-1}}.$$

Solution:

(1) Check absolute convergence first! Take the absolute value:

$$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \left| \frac{(-1)^n}{\sqrt[6]{n^6-1}} \right| = \sum_{n=2}^{\infty} \frac{1}{\sqrt[6]{n^6-1}}.$$

Use Limit Comparison with harmonic series. That is, take $\sum b_n = \sum \frac{1}{n}$, where the harmonic series is divergent. We also have that $a_n, b_n > 0$ for $n \geq 2$. Now take the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1}{\sqrt[6]{n^6-1}} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{\sqrt[6]{n^6-1}} \\
&= \lim_{n\to\infty} \frac{n}{\sqrt[6]{n^6-1}} \cdot \frac{\frac{1}{n}}{\frac{1}{\sqrt[6]{n^6}}} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt[6]{1-\frac{1}{n^6}}} = 1 > 0.
\end{aligned}$$

Since the harmonic series is divergent, the above series diverges. Therefore, the series **does not** converge absolutely.

(2) Now use Alternating Series Test:

(a) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt[6]{n^6-1}} = 0$

(b) Now we must show that b_n is decreasing. Take the derivative

$$f(x) = \frac{1}{\sqrt[6]{x^6-1}} \quad \Rightarrow \quad f'(x) = -\frac{x^5}{(x^6-1)^{\frac{7}{6}}},$$

which is negative for $x \in [2, \infty)$, so the b_n are decreasing. Therefore, the series converges by Alternating Series Test. Since the series did not converge absolutely, the series is **conditionally convergent**.

7) (10 pts.) Find the radius of convergence and interval of convergence for the following power series. (*Note*: This is known as the *Bessel function of order 2*.

$$\sum_{n=0}^{\infty} \frac{(-1)^n \, x^{2(n+1)}}{n! \, (n+2)! \, 2^{2(n+1)}}$$

<u>Solution</u>:

Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+4}}{(n+1)!(n+3)!2^{2n+4}} \cdot \frac{n!(n+2)!2^{2n+2}}{(-1)^n x^{2n+2}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{2n+4}}{x^{2n+2}} \right| \cdot \frac{2^{2n+2}}{2^{2n+4}} \cdot \frac{n!(n+2)!}{(n+3)(n+2)!(n+1)n!} \\
&= \frac{|x|^2}{2^2} \lim_{n\to\infty} \frac{1}{(n+3)(n+1)} = 0 \qquad \text{for all } x.
\end{aligned}$$

The limit being equal to 0 implies that the radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$.

8) (10 pts.) Find the Taylor polynomial of degree 3, centered at the point $a = 0$ for the function $f(x) = e^{-x}$ using the definition of Taylor series. **Note: Do NOT use the substitution method, as you will receive no credit. You must use the definition.**

Solution:

To use the definition of Taylor series, we use the formula on the front of the exam

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For our question, $f(x) = e^{-x}$, $a = 0$, and we are computing $T_3(x)$. Recall that this means there are 4 terms to compute. We need to compute the coefficients which are generated using the derivatives of $f(x)$. So we get

$$\begin{aligned}
f^{(0)}(x) &= e^{-x} \quad \Rightarrow \quad f^{(0)}(0) = 1 \\
f^{(1)}(x) &= -e^{-x} \quad \Rightarrow \quad f^{(1)}(0) = -1 \\
f^{(2)}(x) &= e^{-x} \quad \Rightarrow \quad f^{(2)}(0) = 1 \\
f^{(3)}(x) &= -e^{-x} \quad \Rightarrow \quad f^{(3)}(0) = -1
\end{aligned}$$

Now we can compute the whole coefficient by dividing by the $n!$ to get

$$\begin{aligned}
\frac{f^{(0)}(0)}{0!} &= 1 \\
\frac{f^{(1)}(0)}{1!} &= -1 \\
\frac{f^{(2)}(0)}{2!} &= \frac{1}{2} \\
\frac{f^{(3)}(0)}{3!} &= -\frac{1}{6}
\end{aligned}$$

Therefore, the Taylor series approximation of $f(x)$ is

$$f(x) = e^{-x} \approx T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(a)}{n!}(x-a)^n = 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$$

9) (5 pts.) (a) Compute the following integral using Taylor series.

$$\int e^{-x^2}\,dx$$

(5 pts.) (b) Find the Taylor series centered at $a = 0$ for the function

$$f(x) = \frac{\arctan(x) - x}{x^2}$$

Solution:

(a) We know from the table on the front (line 1 below)

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{-x^2} &= \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \\ \int e^{-x^2}\,dx &= \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \right) dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int x^{2n}\,dx \right) \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} \end{aligned}$$

(b) Use the formula for the Taylor Series centered at 0 from the front page of the test:

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Compute the numerator first by writing out the terms for the above series

$$\begin{aligned} \arctan(x) - x &= \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) - x = -\frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}. \end{aligned}$$

Now multiplying the above series by $\frac{1}{x^2}$, we get the desired Taylor Series

$$\begin{aligned} \frac{\arctan(x) - x}{x^2} &= \frac{1}{x^2} \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\ &= \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n-1}}{2n+1}. \end{aligned}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST