MATH 009C **MIDTERM** **July 12, 2018**

Name: ____________________________ **Score:** _______ / 100

Student ID: ____________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	Total
✓						
Score						
Pts. Possible	10	10	10	10	10	45

INSTRUCTIONS FOR STUDENTS

- You can use both sides of the paper for your solution. This is an **4** question exam.
- Students have **50** minutes to complete the exam.
- The test will be out of **40** points (4 questions). You may attempt a 5^{th} question, which will have a maximum of 5 possible points. The highest possible score is therefore **45** points.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing. Write a ✓ for the problems you want to be graded for credit, and EC for the extra credit problem.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The last page of the test can be used for scratch work.

GOOD LUCK!

1) (8 pts.) (a) Eliminate the parameter for the following parametric equations

$$x = 2\sin(t) \quad y = 5\cos(t)$$

(2 pts.) (b) Identify the type of graph from your result in part (a).

Solution:

(a) Rewrite x and y in terms of $\cos^2(t)$ and $\sin^2(t)$, so we can use the trigonometric identity:

$$\begin{aligned} x &= 2\sin(t) \\ x^2 &= 4\sin^2(t) \\ \frac{x^2}{4} &= \sin^2(t) \\ &\text{and} \\ y &= 5\cos(t) \\ y^2 &= 25\cos^2(t) \\ \frac{y^2}{25} &= \cos^2(t) \end{aligned}$$

By adding the two equations, then using the identity $\sin^2(t) + \cos^2(t) = 1$:

$$\frac{x^2}{4} + \frac{y^2}{25} = 1$$

(b) The graph is an ellipse centered at $(0, 0)$ with major axis $b = \sqrt{25} = 5$ and minor axis $a = \sqrt{4} = 2$.

2) Consider the parametric equations defined as: $x = \sin(t),\ y = -\cos(t)$, for $0 \leq t \leq 2\pi$.

(3 pts.) (a) Compute $\dfrac{dy}{dx}$.

(3 pts.) (b) Compute $\dfrac{d^2y}{dx^2}$.

(4 pts.) (c) For which values of t is the curve concave upward? For which values of t is the curve concave downward?

Solution:

(a) Use the formula for the derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sin(t)}{\cos(t)} = \tan(t)$$

(b) Use the formula for the second derivative:

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\tan(t)}{\cos(t)} = \frac{\sec^2(t)}{\cos(t)} = \frac{1}{\cos^3(t)}$$

(c) For concave upward, we need to find where $\dfrac{d^2y}{dx^2} > 0$. We solve the inequality

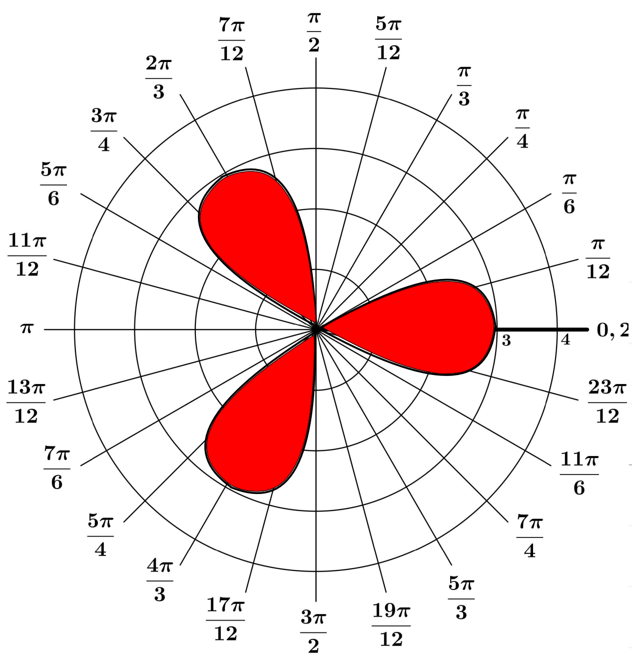
$$\begin{aligned} \frac{d^2y}{dx^2} &> 0 \\ \frac{1}{\cos^3(t)} &> 0 \\ \frac{1}{\cos(t)} &> 0 \\ \cos(t) &> 0 \end{aligned}$$

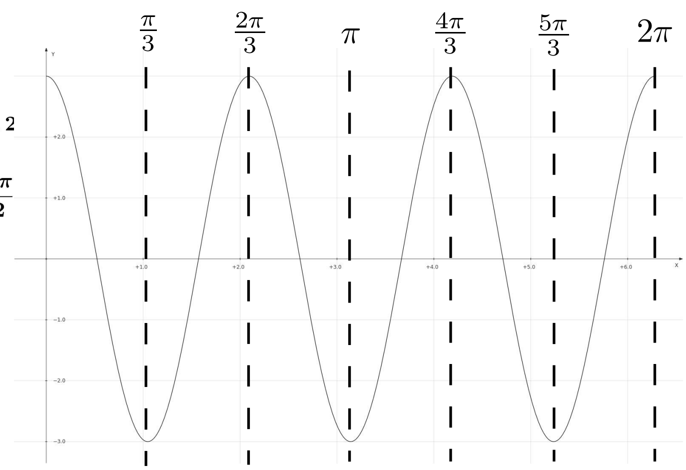
So $\cos(t) > 0$ when $t \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$. For concave down, the inequality sign is reversed, and we get the curve is concave down for $t \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

3) (5 pts.) (a) Sketch the following polar curve on the polar graph paper below:

$$r = 3\cos(3\theta) \qquad 0 \leq \theta \leq \pi$$

(5 pts.) (b) Consider the polar curve from part (a). Find the slope of the tangent line to the curve at the point $\theta = \dfrac{\pi}{2}$.





Solution:

(a) See the diagrams above. We can split the domain $[0, \pi]$ into 3 pieces, so that each piece is $\dfrac{\pi}{3}$ in length, which gives us 3 petals. If we split the interval into 6 pieces, we get 6 half petals, and we can determine that wherever $r < 0$, we will reflect across the origin, to draw the picture.

(b) Use the formula for the derivative:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} = \frac{-9\sin(3\theta)\sin(\theta) + 3\cos(3\theta)\cos(\theta)}{-9\sin(\theta)\cos(\theta) - 3\cos(3\theta)\sin(\theta)}$$

$$\overset{\theta=\frac{\pi}{2}}{=} \frac{-9\cdot(-1)\cdot 1 + 3\cdot 0\cdot 0}{-9\cdot 1\cdot 0 - 3\cdot 0\cdot 1} = \frac{9}{0} = \infty$$

So the tangent line is vertical at $\theta = \dfrac{\pi}{2}$.

Alternatively, since $x = r\cos(\theta) = 3\cos(3\theta)\cos(\theta)$ and $y = r\sin(\theta) = 3\cos(3\theta)\sin(\theta)$, then we use the parametric equations formulas:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-9\sin(3\theta)\sin(\theta) + 3\cos(3\theta)\cos(\theta)}{-9\sin(\theta)\cos(\theta) - 3\cos(3\theta)\sin(\theta)}$$

$$\overset{\theta=\frac{\pi}{2}}{=} \frac{-9\cdot(-1)\cdot 1 + 3\cdot 0\cdot 0}{-9\cdot 1\cdot 0 - 3\cdot 0\cdot 1} = \frac{9}{0} = \infty$$

4) Consider the polar curve $r = 2\cos(\theta)$ for $0 \le \theta \le \pi$.

(5 pts.) (a) Find the values of θ where the tangent line is horizontal for $0 \le \theta < \pi$.
(5 pts.) (b) Find the values of θ where the tangent line is vertical for $0 \le \theta < \pi$.

NOTE: *You do not need to consider the $\frac{0}{0}$ case. If a value of θ gives $\frac{dy}{dx} = \frac{0}{0}$, state that the slope of the tangent line is indeterminate.*

Solution:

Since we need the derivative for both parts, we compute that first. Since

$$x = r\cos(\theta) = 2\cos(\theta)\cos(\theta) = 2\cos^2(\theta)$$
$$y = r\sin(\theta) = 2\cos(\theta)\sin(\theta) = \sin(2\theta)$$

Then we use the parametric equations formulas:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos(2\theta)}{-4\cos(\theta)\sin(\theta)} = \frac{2\cos(2\theta)}{-2\sin(2\theta)} = -\cot(2\theta)$$

(a) The tangent line is horizontal where $\dfrac{dy}{dx} = 0$. So we solve

$$\begin{aligned} -\cot(2\theta) &= 0 \\ \frac{\cos(2\theta)}{\sin(2\theta)} &= 0 \\ \cos(2\theta) &= 0 \\ \cos(u) &= 0 \\ u = \frac{\pi}{2}, \frac{3\pi}{2} \quad &\Rightarrow \quad \theta = \frac{\pi}{4}, \frac{3\pi}{4} \end{aligned}$$

So we have a horizontal tangent line at $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}$.

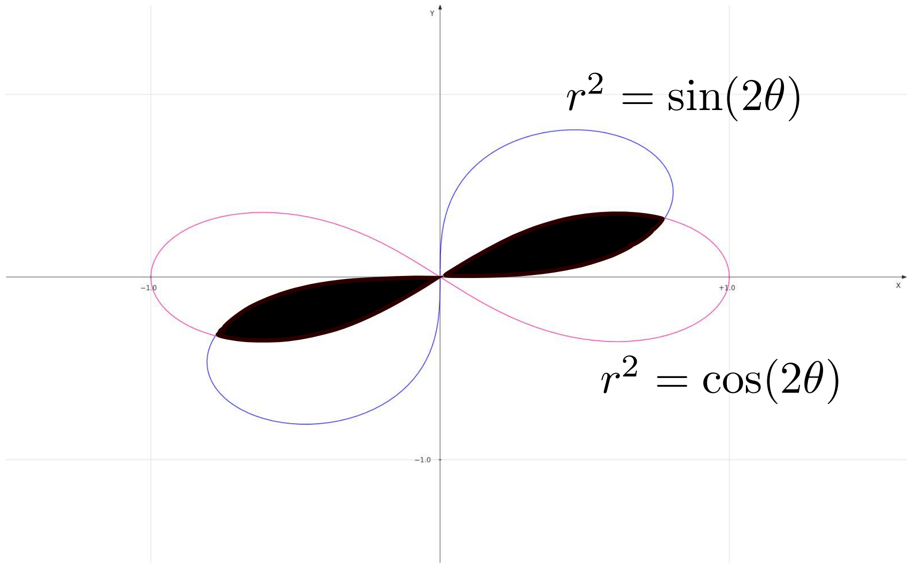
(b) The tangent line is vertical where $\dfrac{dy}{dx} = \pm\infty$. Solving the denominator of the derivative formula equal to zero, we have

$$\begin{aligned} -\cot(2\theta) &= \pm\infty \\ \frac{\cos(2\theta)}{\sin(2\theta)} &= \pm\infty \\ \sin(2\theta) &= 0 \\ \sin(u) &= 0 \\ u = 0, \pi \quad &\Rightarrow \quad \theta = 0, \frac{\pi}{2} \end{aligned}$$

So we have a vertical tangent line at $\theta = 0, \dfrac{\pi}{2}$.

5) (10 pts.) Find the area of the region that lies between the petals (the region is shaded in the labeled plot below).

$$r^2 = \cos(2\theta)$$
$$r^2 = \sin(2\theta)$$



Solution:

Find the intersection point of the curves first. We have to solve the equation:

$$\begin{aligned} \sin(2\theta) &= \cos(2\theta) \\ \tan(2\theta) &= 1 \\ 2\theta &= \frac{\pi}{4} \\ \theta &= \frac{\pi}{8} \end{aligned}$$

There are 2 petals easily visible, but by cleverly choosing the second function, which starts at the origin at $\theta = 0$, from $\theta = 0$ to $\theta = \pi/8$, we get half of the petal in quadrant I. Since the polar rose is symmetric, we really have 4 half petals. Now we compute

$$\begin{aligned} A &= 2 \cdot 2 \int_0^{\pi/8} \frac{1}{2} r^2 \, d\theta \\ &= 4 \int_0^{\pi/8} \frac{1}{2} \sin(2\theta) \, d\theta \\ &= 2 \int_0^{\pi/8} \sin(2\theta) \, d\theta \\ &= -\cos(2\theta)|_0^{\pi/8} \\ &= 1 - \frac{\sqrt{2}}{2} \end{aligned}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST