# Review: Limits, Indeterminate Forms + Improper Integrals



The point of the review is to go over important concepts from 009A and 009B that will be very helpful for this course.

Recall the 2 examples usually shown:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{0}{0} \quad \text{and} \quad \lim_{x \to \infty} \frac{x+1}{x-1} = \frac{\infty}{\infty}$$

These expressions are indeterminate forms, $\frac{0}{0}$ and $\frac{\infty}{\infty}$.
Indeterminate forms in themselves tell us nothing!
The limit could be $0$, $\infty$, or any finite number.
More work is required to find the value of the limit.

Indeterminate Forms: $0/0$, $\frac{\infty}{\infty}$, $\infty \cdot 0$, $\infty - \infty$, $0^0$, $1^\infty$, $\infty^0$

Classic 009A way to deal with indeterminate forms

L'Hôpital's Rule: $f, g$ differentiable, $g'(x) \neq 0$. Suppose $f,g$ diff on $(\alpha, \beta)$ except maybe $a \in (\alpha, \beta)$

① $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$, then

② $\lim\limits_{x \to a} f(x) = \pm\infty$ and $\lim\limits_{x \to a} g(x) = \pm\infty$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if limit on RHS is $\pm\infty$ or limit exists.

Note: We can only use L'Hopital's Rule for $\frac{0}{0}$ or $\frac{\infty}{\infty}$ cases!! The other indeterminate forms must be transformed to one of these to use L'H.

Examples: $\left(\frac{d}{dx}\sec^2 x = 2\sec^2(x)\tan(x)\right)$

① $\lim_{x\to 1} \frac{\ln(x)}{x-1}$

② $\lim_{x\to\infty} \frac{e^x}{x^2}$

③ $\lim_{x\to\infty} \frac{\tan(x)-x}{x^3}$

④ $\lim_{x\to 0^+} x\ln x$

⑤ $\lim_{x\to(\frac{\pi}{2})^-} \sec(x)-\tan(x)$

⑥ $\lim_{x\to 0^+} \left(1+\sin(4x)\right)^{\cot(x)}$

⑦ $\lim_{x\to 0^+} x^x$

⑧ $\lim_{x\to\infty} x^3 e^{-x^2}$

Solution ⑥ $\lim_{x\to 0^+} (1+\sin(4x))^{\cot(x)} = 1^\infty$ indeterminate

$$= \lim_{x\to 0^+} e^{\ln\left[(1+\sin(4x))^{\cot(x)}\right]}$$

$$= \lim_{x\to 0^+} e^{\cot(x)\ln\left[(1+\sin(4x))\right]} = e^{\lim_{x\to 0^+} \frac{\ln(1+\sin(4x))}{\tan(x)}}$$

$$= e^{\lim_{x\to 0^+} \frac{\frac{4\cos(4x)}{1+\sin(4x)}}{\sec^2(x)}} = e^{\lim_{x\to 0^+} \frac{4\cos(4x)}{(1+\sin(4x))}\cdot\frac{\cos^2(x)}{1}}$$

$$= e^4$$

Solution ⑦ $\lim_{x\to 0^+} x^x = 0^0$ indeterminate

Let $y = x^x \Rightarrow \ln(y) = \ln(x^x) = x\ln(x)$

$$\Rightarrow \lim_{x\to 0^+} \ln(y) = \lim_{x\to 0^+} x\ln(x)$$

$$\Rightarrow e^{\lim_{x\to 0^+} \ln(y)} = e^{\lim_{x\to 0^+} x\ln x}$$

$$\lim_{x\to 0^+} y = e^0 = \boxed{1}$$

# Review - Improper Integrals

There are two types of integrals (improper) that we consider in 009B, ones with infinite ($\pm\infty$) bounds and ones with finite bounds and discontinuities in the integrand. We will only consider the former.

Recall we always rewrite these in terms of limits:

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx \quad \text{provided the limit exists}$$

Then if possible, we compute the integral, then take a limit.

Examples: ① $\int_1^\infty \frac{1}{x}\,dx$  ③ $\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

② $\int_{-\infty}^0 xe^x\,dx$  ④ $\int_1^\infty \frac{1}{x^p}\,dx$ (for which $p$?!) (conv. $p>1$, div $p\le 1$)

Comparison Theorem: What if we cannot do the integration directly? We need another method.

Suppose $f, g$ are continuous s.t. $f(x) \geq g(x)\ (\geq 0)\ \forall x \geq a$

① If $\int_a^\infty f(x)\,dx$ converges $\Rightarrow \int_a^\infty g(x)\,dx$ is convergent

② If $\int_a^\infty g(x)\,dx$ diverges $\Rightarrow \int_a^\infty f(x)\,dx$ is divergent

Ex) ① $\int_1^\infty e^{-x^2}\,dx$ convergent or divergent? Compare with $e^{-x}$ (can we integrate? See 11.4)

② $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ (show $\frac{1}{x} < \frac{1+e^{-x}}{x}$)

③ $\int_0^1 \frac{\sec^2 x}{x\sqrt{x}}\,dx$ ($\sec(x) > 1$ for $0<x<1$ $\Rightarrow \sec^2(x) > 1$) compare with $\frac{1}{x^{3/2}}$