Where do we go from here

Applications / Higher Math:

① ODE's → Series Solutions
② PDE's → Series Solutions
③ Fourier Series
④ Probability
⑤ Euler's Formula

① We've seen this last time

② $u_t = u_{xx}$ ⇒ solution $u(x,t)$ is a Fourier Series

$$\frac{\partial}{\partial t}u = \frac{\partial^2}{\partial x^2}u \Rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

$u(0,t) = u(1,t) = 0$

$u(0,x) = f(x)$

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx$$

③ $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$ L const.

$i = \sqrt{-1}$
$\Rightarrow i^2 = -1$
$i^3 = -i$
$i^4 = 1$

④ Problem on next page

⑤ Euler's Formula: $e^{it} = \cos(t) + i\sin t$, $i = \sqrt{-1}$

$$\sin(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \quad \cos(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} \quad e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$$

$$\Rightarrow e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \frac{1}{1} + \frac{it}{1} + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \frac{(it)^5}{5!} + \frac{(it)^6}{6!} + \frac{(it)^7}{7!} + \frac{(it)^8}{8!} + \dots$$

$$= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} + \dots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} \cancel{+} - \frac{t^7}{7!} + \dots\right)$$

$$= \cos(t) + i\sin(t) \checkmark$$

Probability

Q: If one throws two dice one time, the probability of getting a roll of 7 is $p = \frac{1}{6}$ (really its $\frac{6}{36}$). If you throw the dice repeatedly, probability that a 7 appears for the first time on the n^{th} roll is $q^{n-1}p$, where $q = 1 - p = \frac{5}{6}$. The expected number of rolls until the 1^{st} 7 is given as $\sum_{n=1}^{\infty} n q^{n-1} p$. Find the sum.

Note: How did they ged the sum? $\mathbb{E}(X) = \sum x_i p_i$ is expected value (mean, avg.)

We have $n-1$ failures and 1 success on the n^{th} roll. n rolls, q^{n-1}, p

$$\Rightarrow \mathbb{E}\left(\begin{matrix}\text{number of rolls}\\ \text{until first 7}\end{matrix}\right) = \sum_{n=1}^{\infty} n q^{n-1} p$$

Solution: Note that the partial sums are given as (ignore the $p = \frac{1}{6}$ for the moment)

$$S_n = \sum_{j=1}^{n} j\left(\frac{5}{6}\right)^{j-1}$$

This looks geometric. We know $\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r} = \sum_{n=0}^{\infty} a r^n$

Recall, I proved that $\sum_{j=1}^{n} a r^{j-1} = \frac{a(1-r^n)}{1-r}$ n^{th} partial sum of a geometric series

$$\Rightarrow \sum_{j=0}^{n-1} a r^j = \frac{a(1-r^n)}{1-r} \Rightarrow \sum_{j=0}^{n} a r^j = \frac{a(1-r^{n+1})}{1-r}$$

Assume for the moment that $a = 1$

$$\Rightarrow \frac{d}{dr}\sum_{j=0}^{n} r^j = \frac{d}{dr}\left(\frac{(1-r^{n+1})}{(1-r)}\right) \quad \text{product rule}$$

$$\Rightarrow \sum_{j=1}^{n} j r^{j-1} = \frac{1-r^{n+1}}{(1-r)^2} - \frac{(n+1)r^n}{1-r}, \quad \text{Let } r = \frac{5}{6}$$

$$\Rightarrow \underbrace{\sum_{j=1}^{n} j\left(\frac{5}{6}\right)^{j-1}}_{\Vert \; S_n \, (!!)} = \frac{1-(5/6)^{n+1}}{(1-5/6)^2} - \frac{(n+1)(5/6)^n}{1-5/6} = \frac{1-(5/6)^{n+1}}{1/36} - \frac{(n+1)(5/6)^n}{1/6}$$

$$= 36\left(1-(5/6)^{n+1}\right) - 6(n+1)\left(\frac{5}{6}\right)^n$$

So, we have shown the nth partial sum of $\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}$ is given by $S_n = 36\left(1-\left(\frac{5}{6}\right)^{n+1}\right) - 6(n+1)\left(\frac{5}{6}\right)^n$

So, then the sum we want is just $\lim_{n\to\infty} S_n$,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 36\left(1-\left(\frac{5}{6}\right)^{n+1}\right) - 6(n+1)\left(\frac{5}{6}\right)^n$$

$$= 36(1-0) - 6\cdot 0$$

$$= 36$$

$$\lim_{n\to\infty}\left(\frac{5}{6}\right)^{n+1} = 0$$

geometric

$r = \frac{5}{6} < 1$

$$\lim_{n\to\infty}(n+1)\left(\frac{5}{6}\right)^n = \lim_{n\to\infty}\frac{n+1}{\left(\frac{5}{6}\right)^{-n}} = \lim_{n\to\infty}\frac{n+1}{\left(\frac{6}{5}\right)^n} \overset{L'H}{=} \lim_{n\to\infty}\frac{1}{\left(\frac{6}{5}\right)^n\log\left(\frac{6}{5}\right)} = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1} = 36$$

$$\Rightarrow p\sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1} = p\cdot 36$$

$$\Rightarrow \sum_{n=1}^{\infty} n\left(\frac{5}{6}\right)^{n-1}\left(\frac{1}{6}\right) = \frac{1}{6}\cdot 36 = 6$$

$\Rightarrow$ Expected (or average) # of rolls until first 7 is 6.