Section 10.3 - Integral Test

Integral Test: Suppose f is a function that is

(1) continuous, (2) positive, (3) decreasing on $[1,\infty)$

with the property $a_n = f(n)$. There are two cases

$\boxed{1}$ If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

$\boxed{2}$ If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

The proof follows from looking at lower and upper Riemann Sums.
We will not explore this proof for our course.

Note: We can start the interval at any integer, not just 1. We need f to be decreasing "eventually". Recall, a finite number of terms do not affect convergence.

Examples: $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$, $\sum_{n=1}^{\infty} ne^{-n^2}$, $\sum_{n=1}^{\infty} \frac{1}{2^{\ln(n)}}$, $\sum_{n=4}^{\infty} \frac{1}{(n-3)^3}$

p-Series: p-series have the form $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for some constant p.

We must consider the cases $p<0, p=0, p>0, p=1$

If $p<0$, then $\lim_{n\to\infty} \frac{1}{n^p} = \infty \Rightarrow$ by Divergence Test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent

If $p=0$, then $\lim_{n\to\infty} \frac{1}{n^0} = \lim_{n\to\infty} 1 = 1 \Rightarrow$ by Divergence Test $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent.

If $p=1$ or $p>0$, apply Integral Test. $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing for $p>0$ (check!)

$$\int_1^\infty \frac{1}{x^p}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^p}\,dx = \lim_{t\to\infty} \begin{cases} \frac{t^{1-p}}{1-p} + \frac{1}{p-1} & p\neq 1, p>0 \\ \ln(t) & p=1, (p>0) \end{cases}$$

$$= \begin{cases} \frac{1}{p-1} & p>1 \\ \infty & 0<p\leq 1 \end{cases} \Rightarrow$$

$\boxed{\text{p-series are convergent for } p>1 \text{, divergent for } p\leq 1}$