Section 10.4 - Comparison Tests

(Direct) Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. There are two cases

$\boxed{1}$ If $\sum b_n$ is convergent and $a_n \le b_n$ for all n, then $\sum a_n$ is convergent.

$\boxed{2}$ If $\sum b_n$ is divergent and $a_n \ge b_n$ for all n, then $\sum a_n$ is divergent.

To use the comparison test, we need to determine a series $\sum b_n$ which we know is convergent (case $\boxed{1}$) or divergent (case $\boxed{2}$).

We generally try either ① A p-series $\sum \frac{1}{n^p} \rightarrow \begin{cases} p>1 & \text{convergent} \\ p\le 1 & \text{divergent} \end{cases}$

② A geometric series $\sum ar^{n-1} \rightarrow \begin{cases} |r|<1 & \text{convergent} \\ |r|\ge 1 & \text{divergent} \end{cases}$

Examples: $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$, $\sum_{n=1}^{\infty} \frac{1}{2n^2+4n+3}$, $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$, $\sum_{n=2}^{\infty} \frac{n^3}{n^4+1}$

What about $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$? Inequality fails. $\frac{1}{2^n-1} > \frac{1}{2^n}$.

Limit Comparison Test: Suppose that $\sum a_n$ and $\sum b_n$ are two series with positive terms. There are 3 possibilities

$\boxed{1}$ If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, $c<\infty$ and positive, then both series converge or both series diverge.

$\boxed{2}$ If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, and $\sum b_n$ is convergent, then $\sum a_n$ converges

$\boxed{3}$ If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, and $\sum b_n$ is divergent, then $\sum a_n$ is divergent.

Examples: $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$, $\sum_{n=1}^{\infty} \frac{2n^2+3n+7}{\sqrt{n^5+3n^4+5n^2+1}}$, $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}$, $\sum_{n=1}^{\infty} \frac{1}{n!}$ ⊛

Bounds: $\ln(n) < n$

general: $\ln(n) < n^c$ for any $c>0$

⊛ Use $n! = (n)(n-1)\cdots(3)(2) \ge 2\cdot 2\cdots 2 = 2^{n-1}$

$\Rightarrow \frac{1}{n!} \le \frac{1}{2^{n-1}}$, use $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ (or LCT).