MATH 009C **PRACTICE FINAL** **August 11, 2018**

Name: ______________________________ **Score:** _______ / 100

Student ID: __________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	6	7	8	9	Total
✓										80
Score										
Pts. Possible	10	10	10	10	10	10	10	10	10	85

INSTRUCTIONS FOR STUDENTS

- Questions are on both sides of the paper. This is an 9 question exam.
- Students have 2 hours to complete the exam.
- The test will be out of **80** points (8 questions). You may attempt a 9^{th} question, which will have a maximum of 5 possible points. The highest possible score is therefore **85** points.
- In the row with the ✓, mark with a ✓ the problems you want graded for credit, and **EC** for your extra credit problem. **If you do not mark the boxes, problems 1-8 will be graded for credit regardless of which ones you complete, and 9 will be your extra credit.**
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. **Any unjustified claims will receive no credit. This means you need to state which test you are using for series questions!** Clearly box your final answer.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The back of the test can be used for scratch work.

GOOD LUCK!

FORMULAS:

Common Taylor Series	Common Taylor Series
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for all } \lvert x\rvert < 1$	$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all } x \in \mathbb{R}$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all } x \in \mathbb{R}$	$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for all } x \in \mathbb{R}$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad \text{for } x \in (-1, 1]$	$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad \text{for } \lvert x\rvert \le 1$
$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad \text{for } \lvert x-a\rvert < R$	$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n, \quad \text{for } \lvert x\rvert < 1$

1) (5 pts.) (a) Determine whether the sequence converges or diverges:

$$a_n = \frac{(2n-1)!}{(2n+1)!}.$$

(5 pts.) (b) Determine whether the sequence converges or diverges:

$$a_n = \left(1 + \frac{2}{n}\right)^{2n}$$

Solution:

(a) By using factorial properties we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \lim_{n\to\infty} \frac{1}{2n(2n+1)} = 0$$

s So the sequence a_n converges to 0.

(b) Use the exponential-logarithm trick for limits:

$$\begin{aligned} \lim_{x\to\infty} e^{\ln\left(\left(1+\frac{2}{x}\right)^{2x}\right)} &= \lim_{x\to\infty} e^{2x \ln\left(1+\frac{2}{x}\right)} \\ &= e^{\lim_{x\to\infty} \frac{2\ln\left(1+\frac{2}{x}\right)}{1/x}} \\ &= e^{\lim_{x\to\infty} \frac{2\frac{1}{1+\frac{2}{x}} \cdot \frac{-2}{x^2}}{-(1/x^2)}} \\ &= e^{\lim_{x\to\infty} \frac{4}{1+2/x}} = e^4 \end{aligned}$$

where we have applied L'Hopital's rule once. So the sequence converges to e^4.

2) (5 pts.) (a) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \sqrt[n]{2}$$

(5 pts.) (b) Determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{3+4^n}{7^n}.$$

Solution:

(a) Use the Test for Divergence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0.$$

Since the limit is not equal to zero, the series diverges.

(b) The series can be divided into 2 geometric series:

$$\sum_{n=1}^{\infty} \frac{3+4^n}{7^n} = \sum_{n=1}^{\infty} \frac{3}{7^n} + \frac{4^n}{7^n} = \sum_{n=1}^{\infty} \frac{3}{7}\left(\frac{1}{7}\right)^{n-1} + \sum_{n=1}^{\infty} \frac{6}{7}\left(\frac{6}{7}\right)^{n-1},$$

and since both geometric series converge because $1/7 < 1$ and $6/7 < 1$, so the original series is convergent.

3) (10 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{1}{n^2+16}.$$

Solution:

Use the Integral Test. Define $f(x) = \dfrac{1}{x^2+16}$. We have $a_n = f(n)$, and

- $f(x)$ is continuous on $(-\infty, \infty)$ as it is defined for all x; the denominator is never 0.
- $f(x)$ is positive as the denominator is positive for any x.
- $f'(x) = \dfrac{-2x}{(x^2+16)^2} < 0$ for $x > 0 \quad \Rightarrow \quad f(x)$ is decreasing on $[1, \infty)$.

Then compute the improper integral

$$\begin{aligned}
\int_1^{\infty} \frac{1}{x^2+16}\, dx &= \lim_{t\to\infty} \frac{1}{4} \arctan\left(\frac{1}{4}x\right)\Big|_1^t \\
&= \frac{1}{4} \lim_{t\to\infty} \arctan\left(\frac{1}{4}t\right) - \arctan\left(\frac{1}{4}\right) \\
&= \frac{\pi}{16} - \arctan\left(\frac{1}{4}\right).
\end{aligned}$$

So since the improper integral is convergent, the series is convergent by the Integral Test.

4) (10 pts.) Determine whether the series is convergent or divergent

$$\sum_{n=1}^{\infty} \frac{n^3 + 4n + 1}{\sqrt[3]{n^{12} + 2n^4 + n^2 + 1}}$$

Solution:

Use the Limit Comparison Test with $b_n = \dfrac{n^3}{\sqrt[3]{n^{12}}} = \dfrac{n^3}{n^4} = \dfrac{1}{n}$. Note that $a_n > 0$ and $b_n > 0$ for all $n \geq 1$, and $\sum b_n$ is the harmonic series and is divergent. Now take the limit

$$\begin{aligned}
\lim_{n\to\infty} \frac{n^3 + 4n + 1}{\sqrt[3]{n^{12} + 2n^4 + n^2 + 1}} \cdot \frac{n}{1} &= \lim_{n\to\infty} \frac{n^4 + 4n^2 + n}{\sqrt[3]{n^{12} + 2n^4 + n^2 + 1}} \\
&= \lim_{n\to\infty} \frac{n^4 + 4n^2 + n}{\sqrt[3]{n^{12} + 2n^4 + n^2 + 1}} \cdot \frac{\frac{1}{n^4}}{\frac{1}{\sqrt[3]{n^{12}}}} \\
&= \lim_{n\to\infty} \frac{1 + \frac{4}{n^2} + \frac{1}{n^3}}{\sqrt[3]{1 + \frac{2}{n^8} + \frac{1}{n^10} + \frac{1}{n^{12}}}} \\
&= 1 < \infty
\end{aligned}$$

Therefore, since the limit is positive and finite, and the series $\sum \dfrac{1}{n}$ is divergent, the original series diverges by Limit Comparison Test.

5) (5 pts.) (a) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{4^n n!}{(n+2)!}$$

(5 pts.) (b) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$$

Solution:

(a) Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{4^{n+1}(n+1)!}{(n+3)!} \cdot \frac{(n+2)!}{4^n n!} \\
&= \lim_{n\to\infty} \frac{(n+2)!(n+1)!}{(n+3)!n!} \cdot \frac{4^{n+1}}{4^n} \\
&= \lim_{n\to\infty} \frac{(n+2)!(n+1)(n!)}{(n+3)((n+2)!)n!} \cdot \frac{4^{n+1}}{4^n} \\
&= \lim_{n\to\infty} 4\frac{n+1}{n+3} = 4 > 1.
\end{aligned}$$

So the series is divergent by the Ratio Test.

(b) Use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{(n!)^n}{n^{4n}}\right)^{1/n} = \lim_{n\to\infty} \frac{n!}{n^4} = \lim_{n\to\infty} \frac{n(n-1)(n-2)(n-3)(n-4)\ldots}{n \cdot n \cdot n \cdot n} = \infty$$

So the series diverges by the Root Test.

6) (10 pts.) Determine whether the series is absolutely convergent, conditionally convergent, or divergent:

$$\sum_{n=2}^{\infty}(-1)^n \frac{n^n}{n!}$$

Solution:

Note that this is an alternating series. Check absolute convergence: Take absolute value:

$$\sum_{n=2}^{\infty}|a_n| = \sum_{n=2}^{\infty}\left|(-1)^n \frac{n^n}{n!}\right| = \sum_{n=2}^{\infty}\frac{n^n}{n!}.$$

Now use the Ratio Test to check the convergence of this series

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n} \\
&= \lim_{n\to\infty}\frac{(n+1)^{n+1}}{n^n}\cdot\frac{n!}{(n+1)(n!)} \\
&= \lim_{n\to\infty}(n+1)\left(\frac{n+1}{n}\right)^n\cdot\frac{1}{n+1} \\
&= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e > 1.
\end{aligned}$$

Which implies that the series $\sum |a_n|$ is divergent. Therefore, the series $\sum a_n$ **does not** converge absolutely.

Now you try Alternating Series Test. The Alternating Series Test requires that $b_n = \frac{n^n}{n!}$ be decreasing an $\lim\limits_{n\to\infty} b_n = 0$. Both of these statements are false, so Alternating Series Test fails. But if you were to compute the limit you would see the following.

It is actually the Test for Divergence on the original series,

$$\lim_{n\to\infty}(-1)^n\frac{n^n}{n!} = \text{DNE}$$

by using the inequality trick from class (see the Notes!). Therefore, the original series diverges by Divergence Test.

7) (10 pts.) Find the radius of convergence and interval of convergence for the following power series:

$$\sum_{n=0}^{\infty}(-1)^n \frac{(x-2)^n}{3n+1}$$

Solution:

Use the Ratio Test:

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|(-1)^{n+1}\frac{(x-2)^{n+1}}{3(n+1)+1}\cdot\frac{3n+1}{(-1)^n(x-2)^n}\right| \\
&= \lim_{n\to\infty}|x-2|\cdot\frac{3n+1}{3n+4} \\
&= |x-2|\lim_{n\to\infty}\frac{3n+1}{3n+4} = |x-2|.
\end{aligned}$$

From the Ratio Test, if the limit is less than 1, the series converges, so we have $|x-2|<1$, so $R=1$. Solving the inequality, we have that the **tentative** interval of convergence is $1<x<3$. Now we check the endpoints.

$$x=1 \quad\Rightarrow\quad \sum_{n=0}^{\infty}(-1)^n\frac{(-1)^n}{3n+1}=\sum_{n=0}^{\infty}\frac{1}{3n+1} \quad\Rightarrow\quad \text{divergent by LCT with } \sum b_n=\sum\frac{1}{n}$$

$$x=3 \quad\Rightarrow\quad \sum_{n=0}^{\infty}(-1)^n\frac{1^n}{3n+1}=\sum_{n=0}^{\infty}\frac{(-1)^n}{3n+1} \quad\Rightarrow\quad \text{conditionally convergent by AST}$$

Therefore, we have that the radius of convergence is $R=1$, and the interval convergence is $(1,3]$.

8) (10 pts.) Find the Taylor polynomial of degree 3, centered at the point $a = 0$ for the function $f(x) = e^{2x}$ using the definition of Taylor series. **Note: Do NOT use the substitution method, as you will receive no credit. You must use the definition.**

Solution:

To use the definition of Taylor series, we use the formula on the front of the exam

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For our question, $f(x) = e^{2x}$, $a = 0$, and we are computing $T_3(x)$. Recall that this means there are 4 terms to compute. We need to compute the coefficients which are generated using the derivatives of $f(x)$. So we get

$$\begin{aligned}
f^{(0)}(x) &= e^{2x} &\Rightarrow\quad f^{(0)}(0) &= 1\\
f^{(1)}(x) &= 2e^{2x} &\Rightarrow\quad f^{(1)}(0) &= 2\\
f^{(2)}(x) &= 4e^{2x} &\Rightarrow\quad f^{(2)}(0) &= 4\\
f^{(3)}(x) &= 8e^{2x} &\Rightarrow\quad f^{(3)}(0) &= 8
\end{aligned}$$

Now we can compute the whole coefficient by dividing by the $n!$ to get

$$\begin{aligned}
\frac{f^{(0)}(0)}{0!} &= 1\\
\frac{f^{(1)}(0)}{1!} &= 2\\
\frac{f^{(2)}(0)}{2!} &= 2\\
\frac{f^{(3)}(0)}{3!} &= \frac{4}{3}
\end{aligned}$$

Therefore, the Taylor series approximation of $f(x)$ is

$$f(x) = e^{2x} \approx T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(a)}{n!}(x-a)^n = 1 + 2x + 2x^2 + \frac{4}{3}x^3$$

9) (5 pts.) (a) Compute the following integral using Taylor series.

$$\int \arctan(x^2)\ dx$$

(5 pts.) (b) Find the Taylor series centered at $a = 0$ for

$$f(x) = \frac{x - \sin(x)}{x^2}$$

Solution:

(a) We know from the table on the front

$$\begin{aligned}
\arctan(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \\
\arctan(x^2) &= \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1} \\
\int \arctan(x^2)\ dx &= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2n+1}\ dx \\
&= \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{4n+2}}{2n+1}\ dx \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(4n+3)(2n+1)}
\end{aligned}$$

(b) We know from the table on the front

$$\begin{aligned}
\sin(x) &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
x - \sin(x) &= x - \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
&= x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right) \\
&= \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} + \dots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \\
\frac{x - \sin(x)}{x^2} &= \frac{1}{x^2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n+1}}{(2n+1)!} \\
&= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n+1)!}
\end{aligned}$$

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END OF TEST