MATH 009C **PRACTICE MIDTERM** **July 12, 2018**

Name: ____________________________ **Score:** _______ / 100

Student ID: ____________________________

DO NOT OPEN THE EXAM UNTIL YOU ARE TOLD TO DO SO

	1	2	3	4	5	Total
✓						
Score						
Pts. Possible	10	10	10	10	10	45

INSTRUCTIONS FOR STUDENTS

- You can use both sides of the paper for your solution. This is an **4** question exam.
- Students have **50** minutes to complete the exam.
- The test will be out of **40** points (4 questions). You may attempt a 5^{th} question, which will have a maximum of 5 possible points. The highest possible score is therefore **45** points.
- In the above table, the row with the ✓, is for you to keep track of the problems you are attempting/completing. Write a ✓ for the problems you want to be graded for credit, and EC for the extra credit problem.
- You may complete parts of problems, as partial credit will be given based on correctness, completeness, and ideas that are leading to the correct solutions.
- **PLEASE SHOW ALL WORK**. Any unjustified claims will receive no credit.
- No notes, textbooks, phones, calculators, etc. are allowed for the exam.
- The last page of the test can be used for scratch work.

GOOD LUCK!

1) (8 pts.) (a) Eliminate the parameter in the for the following parametric equation

$$x = 9\cos(t) + 1 \quad y = 4\sin(t) + 2$$

(2 pts.) (b) Identify the type of graph from your result in part (a).

Solution:

(a) Rewrite x and y in terms of $\cos^2(t)$ and $\sin^2(t)$, so we can use the trigonometric identity:

$$x = 9\cos(t) + 1$$
$$(x-1)^2 = 81\cos^2(t)$$
$$\frac{(x-1)^2}{81} = \cos^2(t)$$

and

$$y = 4\sin(t) + 2$$
$$(y-2)^2 = 16\sin^2(t)$$
$$\frac{(y-2)^2}{16} = \sin^2(t)$$

By adding the two equations, then using the identity $\sin^2(t) + \cos^2(t) = 1$:

$$\frac{(x-1)^2}{81} + \frac{(y-2)^2}{16} = 1$$

(b) The graph is an ellipse centered at $(1, 2)$ with major axis $a = \sqrt{81} = 9$ and minor axis $b = \sqrt{16} = 4$.

2) Consider the parametric defined as: $x = 4 + t^2, y = t^2 + t^3$

(3 pts.) (a) Compute $\dfrac{dy}{dx}$.

(4 pts.) (b) Compute $\dfrac{d^2y}{dx^2}$.

(3 pts.) (c) For which values of t is the curve concave upward?

Solution:

(a) Use the formula for the derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + 3t^2}{2t} = 1 + \frac{3}{2}t$$

(b) Use the formula for the second derivative:

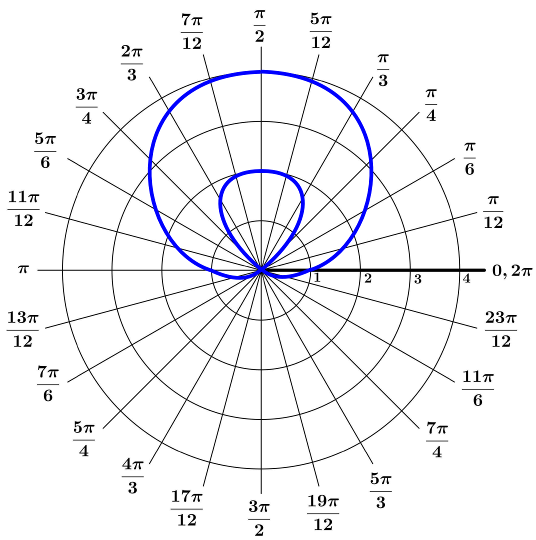
$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}(1 + \frac{3}{2}t)}{2t} = \frac{3}{4t}$$

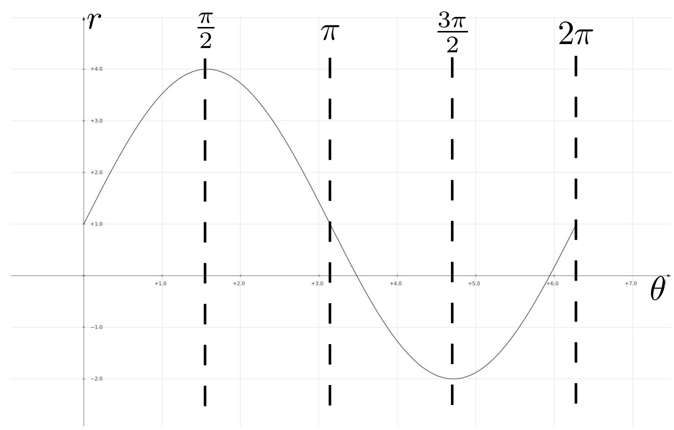
(c) We need to find where $\dfrac{d^2y}{dx^2} > 0$. This is straightforward; since $\dfrac{3}{4t}$ is positive the same place $t > 0$, which is exactly $t > 0$. Therefore, the curve is concave upward for $t > 0$.

3) (5 pts.) (a) Sketch the following polar curve on the polar graph paper below:

$$r = 1 + 3\sin(\theta)$$

(5 pts.) (b) Consider the polar curve from part (a). Find the slope of the tangent line to the curve at the point $\theta = \frac{3\pi}{2}$.





Solution:

(a) See the diagrams above. We can find where $r = 0$ by solving

$$r = 1 + 3\sin(\theta) \Rightarrow 0 = 1 + 3\sin(\theta) \Rightarrow -\frac{1}{3} = \sin(\theta) \Rightarrow \theta = \pi + \arcsin\left(\frac{1}{3}\right)$$

The actual numerical value is not important, as we just need a sketch. Which means that that for $0 \le \theta \le \pi + \arcsin\left(\frac{1}{3}\right)$, the value of r is positive, so we draw the plot in quadrant I, then in quadrant II, and just a small piece in quadrant III. Then, for $\pi + \arcsin\left(\frac{1}{3}\right) \le \theta < 2\pi - \arcsin\left(\frac{1}{3}\right)$, the value of r is negative, so instead of quadrant III, we are in quadrant I. And instead of quadrant IV, we are in quadrant II. Then notice the small piece that becomes positive gives the last part of the graph in quadrant IV.

(b) Use the formula for the derivative:

$$\begin{aligned}\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} &= \frac{3\cos(\theta)\sin(\theta) + (1 + 3\sin(\theta))\cos(\theta)}{3\cos^2(\theta) - (1 + 3\sin(\theta))\sin(\theta)} \\ &= \frac{6\cos(\theta)\sin(\theta) + \cos(\theta)}{3\cos^2(\theta) - 3\sin^2(\theta) - \sin(\theta)} \\ &\overset{\theta=\frac{3\pi}{2}}{=} \frac{6 \cdot 0 \cdot (-1) + 0}{3 \cdot 0 - 3 \cdot (-1)^2 - (-1)} = \frac{0}{-2} = 0\end{aligned}$$

Alternatively, since $x = r\cos(\theta) = (1 + 3\sin(\theta))\cos(\theta) = \cos(\theta) + \frac{3}{2}\sin(2\theta)$ and $y = r\sin(\theta) = (1 + 3\sin(\theta))\sin(\theta) = \sin(\theta) + 3\sin^2(\theta)$. Then we use the parametric equations formulas:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos(\theta) + 6\sin(\theta)\cos(\theta)}{-\sin(\theta) + 3\cos(2\theta)} \overset{\theta=\frac{3\pi}{2}}{=} \frac{0 + 6 \cdot (-1) \cdot (0)}{-(-1) + 3(-1)} = \frac{0}{-2} = 0$$

4) Consider the polar curve $r = 1 + \sin(\theta)$.

(5 pts.) (a) Find the values of θ where the tangent line is horizontal for $0 \le \theta < 2\pi$.
(5 pts.) (b) Find the values of θ where the tangent line is vertical for $0 \le \theta < 2\pi$.

NOTE: *You do not need to consider the $\frac{0}{0}$ case. If a value of θ gives $\frac{dy}{dx} = \frac{0}{0}$, state that the slope of the tangent line is indeterminate.*

Solution:

Since we need the derivative for both parts, we compute that first. Since

$$x = r\cos(\theta) = (1+\sin(\theta))\cos(\theta) = \cos(\theta) + \frac{1}{2}\sin(2\theta)$$
$$y = r\sin(\theta) = (1+\sin(\theta))\sin(\theta) = \sin(\theta) + \sin^2(\theta)$$

Then we use the parametric equations formulas:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos(\theta) + 2\sin(\theta)\cos(\theta)}{-\sin(\theta) + \cos(2\theta)}$$

(a) The tangent line is horizontal where $\frac{dy}{dx} = 0$. So we solve

$$\cos(\theta) + 2\sin(\theta)\cos(\theta) = 0$$
$$\cos(\theta)(1 + 2\sin(\theta)) = 0$$
$$\cos(\theta) = 0 \quad \text{and} \quad \sin(\theta) = -\frac{1}{2}$$
$$\theta = \frac{\pi}{2}, \frac{3\pi}{2} \quad \text{and} \quad \theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

So we have a horizontal tangent line at $\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}$.

(b) The tangent line is vertical where $\frac{dy}{dx} = \infty$. We will need the identity

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

Solving the denominator of the derivative formula equal to zero, we have

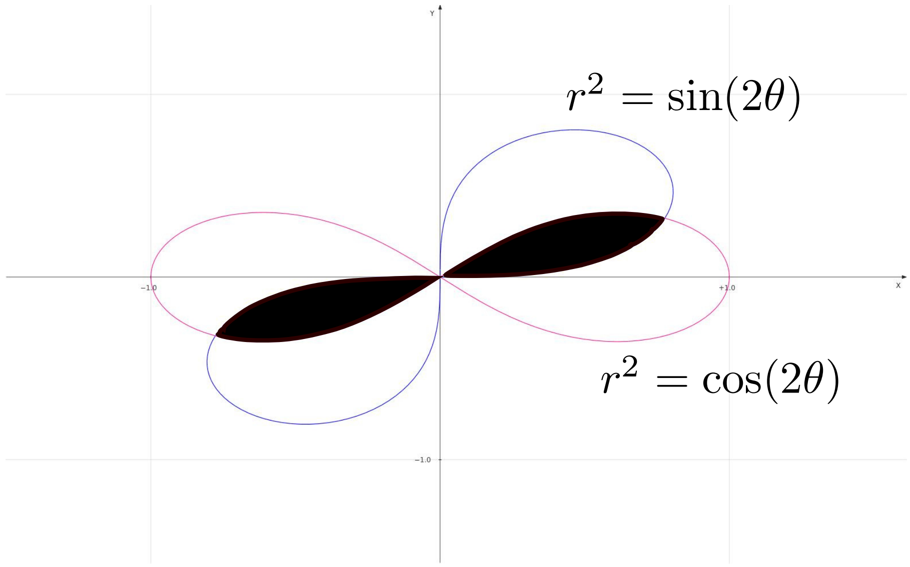
$$-\sin(\theta) + \cos(2\theta) = 0$$
$$-\sin(\theta) + \cos^2(\theta) - \sin^2(\theta) = 0$$
$$2\sin^2(\theta) + \sin(\theta) - 1 = 0 \quad \text{let } u = \sin(x)$$
$$2u^2 + u - 1 = 0$$
$$(2u-1)(u+1) = 0$$
$$(2\sin(\theta) - 1)(\sin(\theta) + 1) = 0$$
$$\theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad \text{and} \quad \theta = \frac{3\pi}{2}$$

where we have used $\alpha = \theta$ and $\beta = \theta$ in the identity to simplify. So we have a vertical tangent line at $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$.

The tangent line is **indeterminate** at $\theta = \frac{3\pi}{2}$

5) (10 pts.) Find the area of the region that lies between the petals (the region is shaded in the labeled plot below).

$$r^2 = \cos(2\theta)$$
$$r^2 = \sin(2\theta)$$



Find the intersection point of the curves first. We have to solve the equation:

$$\sin(2\theta) = \cos(2\theta)$$
$$\tan(2\theta) = 1$$
$$2\theta = \frac{\pi}{4}$$
$$\theta = \frac{\pi}{8}$$

There are 2 petals easily visible, but by cleverly choosing the second function, which starts at the origin at $\theta = 0$, from $\theta = 0$ to $\theta = \pi/8$, we get half of the petal in quadrant I. Since the polar rose is symmetric, we really have 4 half petals. Now we compute

$$\begin{aligned} A &= 2 \cdot 2 \int_0^{\pi/8} \frac{1}{2} r^2 \; d\theta \\ &= 4 \int_0^{\pi/8} \frac{1}{2} \sin(2\theta) \; d\theta \\ &= 2 \int_0^{\pi/8} \sin(2\theta) \; d\theta \\ &= -\cos(2\theta)|_0^{\pi/8} \\ &= 1 - \frac{\sqrt{2}}{2} \end{aligned}$$

THIS PAGE IS LEFT BLANK FOR ANY SCRATCH WORK

END OF TEST