Last Name: First Name:

MATH 009C - Summer 2018

Worksheet 2: July 3, 2018

1. Find an equation of the tangent line at the given point in two ways:
(a) without eliminating the parameter (use the parametric equation formula)
(b) by eliminating the parameter, and then taking the derivative.

$$x = \tan(t) \qquad y = \sec(t) \qquad \text{at the point } (1, \sqrt{2})$$

Can you identify the graph?

Solution:

(a) In order to use the parametric equations formula for $\frac{dy}{dx}$, we need the value of t that gives $(x, y) = (1, \sqrt{2})$. So we solve the system

$$1 = \tan(t) \qquad \sqrt{2} = \sec(t) \quad \Rightarrow \quad t = \frac{\pi}{4}$$

which means $t = \frac{\pi}{4}$ since $(1, \sqrt{2})$ is in the first quadrant. Now we use the parametric equation formula

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\sec(t)}{\frac{d}{dt}\tan(t)} = \frac{\sec(t)\tan(t)}{\sec^2(t)} = \frac{\sin(t)}{\cos(t)} \cdot \frac{\cos(t)}{1} = \sin(t)$$

So the slope at $t = \frac{\pi}{4}$ is $\frac{dy}{dx} = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. Using the point-slope formula, we get the tangent line is $y = \frac{\sqrt{2}}{2}(x-1) + \sqrt{2}$.

(b) Eliminate the parameter similar to the example from lecture

$$x + y = \tan(t) + \sec(t) = \frac{1+\sin(t)}{\cos(t)} \quad ; \quad y - x = \sec(t) - \tan(t) = \frac{1-\sin(t)}{\cos(t)}$$

$$\Rightarrow \quad (x+y)(y-x) = \frac{1+\sin(t)}{\cos(t)} \cdot \frac{1-\sin(t)}{\cos(t)} = \frac{1-\sin^2(t)}{\cos^2(t)} = 1$$

$$\Rightarrow \quad y^2 - x^2 = 1 \qquad \text{hyperbola, opens up/down}$$

Then we use implicit differentiation, and $(x, y) = (1, \sqrt{2})$ to get the slope

$$y^2 - x^2 = 1 \quad \Rightarrow \quad 2y\frac{dy}{dx} - 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and we get the same solution: $y = \frac{\sqrt{2}}{2}(x-1) + \sqrt{2}$.

2. Use parametric equations to show that the area of an ellipse is $A = \pi ab$.

$$x = a\sin(t) \qquad y = b\cos(t) \qquad \text{for } 0 \leq t \leq 2\pi$$

where a and b are constants.

Solution:

Here we can directly use the formula for area of a parametric curve, where $f(t) = a\sin(t)$ and $g(t) = b\cos(t)$, to get

$$\begin{aligned} A &= \int_{\alpha}^{\beta} g(t)f'(t)\ dt \\ &= \int_{0}^{2\pi} b\cos(t) \cdot a\cos(t)\ dt \\ &= ab\int_{0}^{2\pi} \cos^2(t)\ dt \\ &= ab\int_{0}^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2t)\ dt \\ &= ab\left(\frac{1}{2}t + \frac{1}{4}\sin(2t)\right)\Big|_{0}^{2\pi} \\ &= \pi ab \end{aligned}$$

Please, show all work.

3. Find the arc length of the parametric curve

$$x = t^2 \qquad y = t^3$$

between the points (1,1) and (4,8).

Solution:

In order to use the formula, we need the bounds in the parameter t. We are given the points in (x, y) coordinates. By looking at the parametric equations, we see that $(x, y) = (1, 1)$ corresponds to $t = 1$, and $(x, y) = (4, 8)$ corresponds to $t = 2$. Alternatively, you could just solve the system of equations for the appropriate t value, and get the same answer. Now use the formula,

$$\begin{aligned}
L &= \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2}\, dt \\
&= \int_1^2 \sqrt{4t^2 + 9t^4}\, dt \\
&= \int_1^2 t\sqrt{4 + 9t^2}\, dt \qquad \text{let } u = 4 + 9t^2, \text{ and } du = 18t \\
&= \frac{1}{18} \int_{13}^{40} \sqrt{u}\, du \\
&= \frac{1}{18} \cdot \frac{2}{3} \left(u^{\frac{3}{2}}\right)\Big|_{13}^{40} \\
&= \frac{1}{27}\left(40^{\frac{3}{2}} - 13^{\frac{3}{2}}\right)
\end{aligned}$$

Please, show all work.