Last Name: First Name:

MATH 009C - Summer 2018

Worksheet 3: July 10, 2018

1. Give a sketch of the following polar curves. Be sure to label the x and y intercepts of the curves.

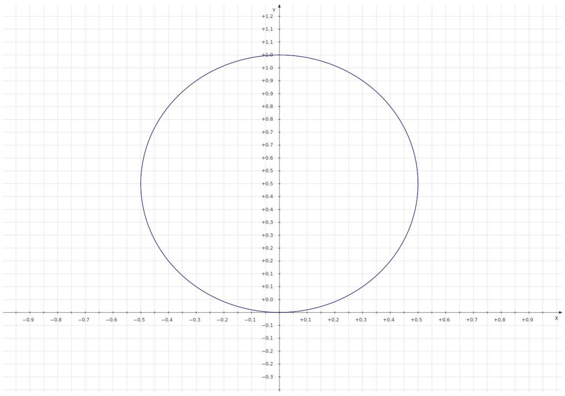
(a) $r = \sin(\theta)$
(b) $r = 4\sin(3\theta)$

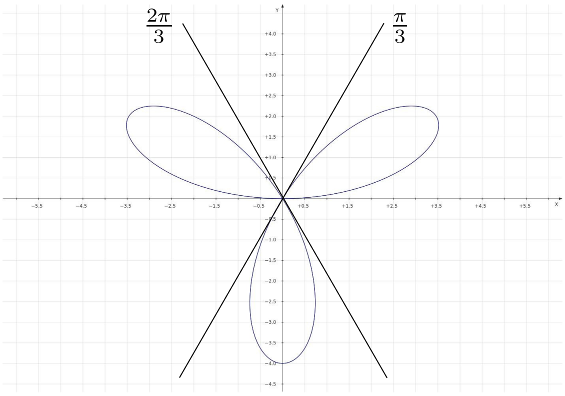
Solution:

(a) Note that we can eliminate the parameter by using polar coordinate formulas:

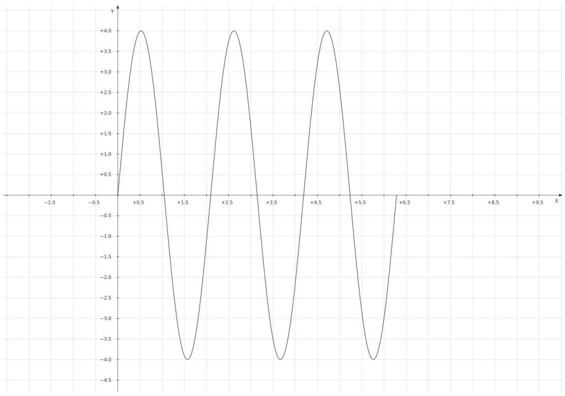
$$\begin{aligned} r = \sin(\theta) \;\Rightarrow\; r^2 = r\sin(\theta) \;\Rightarrow\; x^2 + y^2 = y \\ \Rightarrow\; x^2 + y^2 - y = 0 \;\Rightarrow\; x^2 + y^2 - y + \frac{1}{4} = \frac{1}{4} \\ \Rightarrow\; x^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2 \end{aligned}$$

The graph is a circle of radius $\frac{1}{2}$, with center at $(0, \frac{1}{2})$. It is graphed in the figure on the left below.





(b) For $r = 4\sin(3\theta)$, it is helpful to graph this function in the $r\theta$-parameter plane. The



plot of $r = 4\sin(3\theta)$ for $0 \leq \theta \leq 2\pi$, in the $r\theta$-parameter plane is given directly above, and shows what the radius is for a given θ. Notice that the $r = 0$ points occur at $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$, and 2π. These are the points that correspond to the lines of the graph in xy-space. Notice also that halfway between the $r = 0$ points, we have the maximum and minimums at $r = \pm 4$. These are the tips of the petals in the plot.

2. (a) Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

$$r = 2\sin(\theta), \qquad \theta = \frac{\pi}{6}$$

(b) Find the values of θ for the given polar curve where the tangent line is horizontal and vertical (Restrict to $0 \leq \theta \leq 2\pi$).

$$r = 2\sin(\theta)$$

Solution:

(a) By direct computation:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)} = \frac{2\cos(\theta)\sin(\theta) + 2\sin(\theta)\cos(\theta)}{2\cos^2(\theta) - 2\sin^2(\theta)} \\
&= \frac{4\cos(\theta)\sin(\theta)}{2(\cos^2(\theta) - \sin^2(\theta))} = \frac{2\sin(2\theta)}{2(\cos^2(\theta) - \sin^2(\theta))} \\
&= \frac{2\frac{\sqrt{3}}{2}}{2\left(\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right)} \\
&= \sqrt{3}
\end{aligned}$$

(b) From part (a), we already have the derivative $\dfrac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{2\sin(2\theta)}{2(\cos^2(\theta) - \sin^2(\theta))}$$

First we will find where the numerator and denominator are equal to zero, then consider the various situations.

$$\begin{aligned}
\frac{dy}{d\theta} &= 0 = 2\sin(2\theta) \\
&\Rightarrow\ 0 = \sin(2\theta) \\
&\Rightarrow\ 0 = \sin(u) \qquad \text{for } u = 2\theta \\
&\Rightarrow\ u = 0, \pi, 2\pi, 3\pi, 4\pi \\
&\Rightarrow\ \theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi \\
\frac{dx}{d\theta} &= 0 = 2(\cos^2(\theta) - \sin^2(\theta)) \\
&\Rightarrow\ 0 = \cos^2(\theta) - \sin^2(\theta) \\
&\Rightarrow\ 0 = \cos(2\theta) \qquad \text{using identity } \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta) \\
&\Rightarrow\ 0 = \cos(u) \qquad \text{for } u = 2\theta \\
&\Rightarrow\ u = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \\
&\Rightarrow\ \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}
\end{aligned}$$

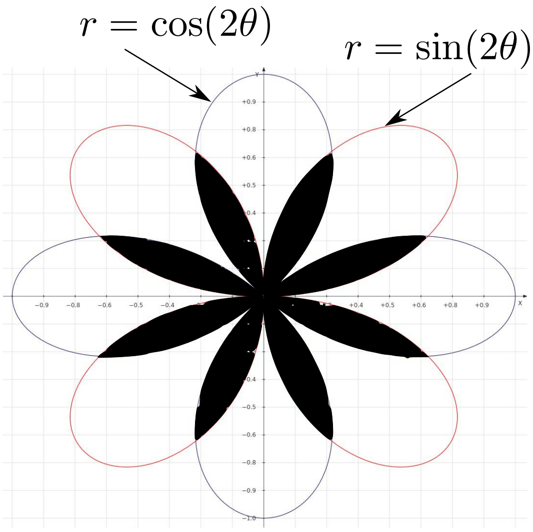
Since none of the values appear in both $\dfrac{dy}{d\theta} = 0$ and $\dfrac{dx}{d\theta} = 0$, we don't have to worry about $\dfrac{0}{0}$. Therefore, we have horizontal tangents at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$, and vertical tangents at $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$.

Please, show all work.

3. Find the area of the region that lies inside the polar roses (the region is shaded in the labeled plot below), for $0 \le \theta \le 2\pi$.

$$r = \cos(2\theta)$$
$$r = \sin(2\theta)$$

Hint: Identities that may be helpful: 1) $\sin^2(2\theta) = \frac{1}{2} - \frac{1}{2}\cos(4\theta)$, *2)* $\cos^2(2\theta) = \frac{1}{2} + \frac{1}{2}\cos(4\theta)$.



__Solution:__

Find the intersection point first. We have to solve the equation:

$$\begin{aligned} \sin(2\theta) &= \cos(2\theta) \\ \tan(2\theta) &= 1 \\ 2\theta &= \frac{\pi}{4}, \frac{5\pi}{4} \\ \theta &= \frac{\pi}{8}, \frac{5\pi}{8} \end{aligned}$$

Note we have 8 petals, but by cleverly choosing the second function, which starts at the origin at $\theta = 0$, from $\theta = 0$ to $\theta = \pi/8$, we get half of the first shaded petal. Since the polar rose is symmetric, really have 16 half petals. Now we compute and use hint 2:

$$\begin{aligned} A &= 8 \cdot 2 \int_0^{\pi/8} \frac{1}{2} r^2 \, d\theta \\ &= 16 \int_0^{\pi/8} \frac{1}{2} \sin^2(2\theta) \, d\theta \\ &= 8 \int_0^{\pi/8} \left(\frac{1}{2} - \frac{1}{2}\cos(4\theta) \right) \, d\theta \\ &= 4 \left(\theta - \frac{1}{4}\sin(4\theta) \right) \Big|_0^{\pi/8} \\ &= \frac{\pi}{2} - 1 \end{aligned}$$

Please, show all work.