LAST NAME: FIRST NAME:

MATH 009C - Summer 2018

Worksheet 4: July 17, 2018

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1. Determine whether the sequence converges or diverges. If it converges, find its limit.

$$(a) \qquad a_n = n \sin\left(\frac{1}{n}\right)$$

$$(b) \qquad a_n = \left(1 + \frac{2}{n}\right)^n$$

$$(c) \qquad a_n = \sin\left(\left(n - \frac{1}{2}\right)\pi\right)$$

<u>Solution</u>:

(a) By rewriting the n in front, and using the substitution $u = \dfrac{1}{n}$, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{u\to 0} \frac{\sin(u)}{u} = 1 \qquad \text{convergent}$$

(b) For this sequence, notice that the n is in the exponent. From the earlier part of the course on L'Hopital's Rule, recall that we use the exponential-logarithm trick to deal with these cases since we have the indeterminate form 1^∞.

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty}\left(1 + \frac{2}{n}\right)^n = \lim_{n\to\infty} \exp\left(\ln\left(\left(1 + \frac{2}{n}\right)^n\right)\right) \\
&= \lim_{n\to\infty} \exp\left(n \ln\left(1 + \frac{2}{n}\right)\right) = \lim_{n\to\infty} \exp\left(\frac{\ln\left(1 + \frac{2}{n}\right)}{\frac{1}{n}}\right) \qquad \text{Use } u = \frac{1}{n} \\
&= \exp\left(\lim_{u\to 0^+} \frac{\ln(1 + 2u)}{u}\right) \qquad \text{Use L'Hopital's Rule} \\
&= \exp\left(\lim_{u\to 0^+} \frac{\frac{2}{1+2u}}{1}\right) = \exp\left(\lim_{u\to 0^+} \frac{2}{1 + 2u}\right) \\
&= \exp(2) = e^2 \qquad \text{convergent}
\end{aligned}$$

(c) Let's write out the some terms in the sequence and see what we can deduce.

$$a_1 = \sin\left(\frac{\pi}{2}\right) = 1, \quad a_2 = \sin\left(\frac{3\pi}{2}\right) = -1, \quad a_3 = \sin\left(\frac{5\pi}{2}\right) = 1$$
$$a_4 = \sin\left(\frac{7\pi}{2}\right) = -1, \quad a_5 = \sin\left(\frac{9\pi}{2}\right) = 1$$

So we can see that the pattern, the terms oscillate from 1 to -1, so we have that

$$a_n = \sin\left(\left(n - \frac{1}{2}\right)\pi\right) = (-1)^n$$

So then we know from lecture that

$$\lim_{n\to\infty} a_n = \sin\left(\left(n - \frac{1}{2}\right)\pi\right) = \lim_{n\to\infty} (-1)^n = \text{DNE} \qquad \text{divergent}$$

Please, show all work.

2. Consider two games, Game A and Game B, with the following rules:

In Game A, you simply lose \$1 every time you play.
In Game B, you count how much money you have left. If it is an even number, you win \$3. Otherwise you lose \$5.

Suppose you start out with \$100. Answer and explain your solutions to the following questions:

- What happens if you just play Game A exclusively?
- What happens if you play Game B exclusively?
- What happens if you play them alternatively, say B, then A, then B, etc. (So that is BABABABA...)? What can you conclude?
- **Challenge**: Can you use sequences to explain the result?

Solution:
Consider playing Game A exclusively. You will obviously lose all your money in 100 rounds. If $X_0 = 100$ is the amount we start with, then we have the following sequence, X_n, which describes the amount of money we have after game n

$$X_n = \{100, 99, 98, 97, 96, \ldots, 3, 2, 1, 0\} = 100 - n$$

where the sequence is finite, as we do not consider negative winnings. Therefore, we go bankrupt in Game A.

Similarly, if you decide to play Game B exclusively, you will also lose all your money in 100 games. Assuming $X_0 = 100$, the sequence will be

$$X_n = \begin{cases} X_{n-1} + 3 & \text{if } X_{n-1} \text{ is even} \\ X_{n-1} - 5 & \text{if } X_{n-1} \text{ is odd} \end{cases}$$

$$= \{100, 103, 98, 101, 96, 99, 94, 97, 92, 95, 90, ...\}$$

We can also see that after any two consecutive games, we lose \$2, by looking at the sequences

$$\begin{aligned} X_{2k} &= X_0 - 2k \\ X_{2k-1} &= X_0 - (2k - 5) \qquad \text{for } k = 1, 2, 3, \ldots \end{aligned}$$

which are subsequences that both decrease as k gets larger, which gives us an indicator of what is going on. We an also look at the difference of our amount of money before and after each game, to get the new sequence

$$Y_n = \{3, -5, 3, -5, 3, -5, 3, -5, 3, -5, 3, ...\}$$

Note that this sequence does not have a limit! But we can interpret what this means. We can write the profit after the n^{th} game as $P_n = X_n - X_{n-1}$ for $n \geq 1$. The total profit after 100 games would be

$$P = \sum_{n=1}^{100} P_n = \sum_{n=1}^{100} X_n - X_{n-1} = 50(3) + 50(-5) = 50(3 - 5) = -100$$

(or see the sum of the terms telescopes to $0 - 100 = -100$) and we started with $X_0 = 100$, therefore, our net gain is $N = X_0 + P = 100 - 100 = 0$. So we will always end up going bankrupt in Game B after 100 games.

Now consider playing the games alternatively, starting with Game B, followed by A, then by B, and so on (BABABA...). Writing out the terms as we have below, we can now see that we will steadily earn a total of \$2 for every two games. In other words, we gain 3 and lose 1. Again, we let $X_0 = 100$, and for $n \geq 1$, we have

$$X_n = \begin{cases} X_{n-1} + 3 & \text{if } n-1 \text{ is even} \\ X_{n-1} - 1 & \text{if } n-1 \text{ is odd} \end{cases}$$

$$= \{100, 103, 102, 105, 104, 107, 106, 109, 108, 111, 110, ...\}$$

This sequence will diverge to infinity, so we could win unlimited money. We can do the same setup as before, but in more generality, so the total profit after M games would be

$$P = \sum_{n=1}^{M} P_n = \sum_{n=1}^{M} X_n - X_{n-1} = \frac{M}{2}(3) + \frac{M}{2}(-1) = \frac{M}{2}(3-1) = M$$

Hence, as the number of games we play, M, goes to infinity, the sum will diverge. This mathematically explains our infinite winnings. So, even though each game is a losing proposition if played alone, because the results of Game B are affected by Game A, the sequence in which the games are played can affect how often Game B earns you money, and subsequently the result is different from the case where either game is played by itself.

Please, show all work.

3. Use the Squeeze Theorem to show that the following sequence is convergent.

$$a_n = \frac{\cos^n(n)}{\ln(n)} \qquad \text{for } n \geq 2$$

<u>**Solution:**</u>
To apply the Squeeze Theorem, we have to come up with two sequences, one which is smaller than (say b_n), and one that is larger than (say c_n) the a_n. We also need the inequality $b_n \leq a_n \leq c_n$ to be true, and

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n.$$

We start off with the cosine first, as it has an immediate set of bounds

$$\begin{aligned} -1 \leq \cos(n) &\leq 1 \\ -1 \leq \cos^n(n) &\leq 1 \\ -\frac{1}{\ln(n)} \leq \frac{\cos^n(n)}{\ln(n)} &\leq \frac{1}{\ln(n)} \end{aligned}$$

since any power of cosine will still be bounded between -1 and 1, and then we can multiply by $\frac{1}{\ln(n)}$, as the logarithm is always positive for any $n \geq 2$, which is the range we are considering here. So far, we have shown the inequality is true. The limits for these two sequences are easy to compute

$$\begin{aligned} \lim_{n\to\infty} b_n &= \lim_{n\to\infty} -\frac{1}{\ln(n)} = 0 \\ \lim_{n\to\infty} c_n &= \lim_{n\to\infty} \frac{1}{\ln(n)} = 0 \end{aligned}$$

Therefore, since we have shown all of the conditions of the Squeeze Theorem, it must be the case that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\cos^n(n)}{\ln(n)} = 0$$

Please, show all work.