LAST NAME: FIRST NAME:

MATH 009C - Summer 2018

Worksheet 5: July 24, 2018

1. For parts (a) and (b), determine whether the series converges or diverges. If the series converges, find its limit. For part(c), find a rational number that gives the decimal expansion.

$$(a) \qquad \sum_{n=1}^{\infty} 2^{1-n}3^{-3n}9^{n-1}$$

$$(b) \qquad \sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$$

$$(c) \qquad 3.\overline{1425} = 3.1425142514251425\ldots$$

<u>Solution</u>:

(a) Notice each piece is exponential, which should signal to us that we will have a geometric series. Now we just make it look like the formula to determine the values of a and r from the formula. We do the algebra and use the rules for exponential functions

$$\begin{aligned}\sum_{n=1}^{\infty} 2^{1-n}3^{-3n}9^{n-1} &= \sum_{n=1}^{\infty} 3^{-3n}\frac{9^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \left(3^{-3}\right)^n \frac{9^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{1}{27}\right)^n \frac{9^{n-1}}{2^{n-1}} \\ &= \sum_{n=1}^{\infty} \frac{1}{27}\left(\frac{1}{27}\right)^{n-1}\frac{9^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \frac{1}{27}\left(\frac{9}{27\cdot 2}\right)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{1}{27}\left(\frac{1}{6}\right)^{n-1}\end{aligned}$$

From here we see that $a = \dfrac{1}{27}$ and $r = \dfrac{1}{6}$, so since $|r| < 1$, the series converges. It's sum can be computed using the formula from lecture

$$\sum_{n=1}^{\infty} \frac{1}{27}\left(\frac{1}{6}\right)^{n-1} = \frac{a}{1-r} = \frac{\frac{1}{27}}{1-\frac{1}{6}} = \frac{2}{45}$$

(b) This is the series corresponding to the sequence formula in number 1 from the previous worksheet. Use Divergence Test:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n\sin\left(\frac{1}{n}\right) = \lim_{n\to\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{u\to 0}\frac{\sin(u)}{u} = 1 \neq 0$$

So since the limit is not equal to zero, by the Divergence Test, the series is divergent.

(c) Group every four digits, as this is when they repeat

$$\begin{aligned}3.\overline{1425} &= 3 + 0.1425 + 0.00001425 + 0.000000001425 + 0.0000000000001425 + \ldots \\ &= 3 + \frac{1425}{10^4} + \frac{1425}{10^8} + \frac{1425}{10^{12}} + \frac{1425}{10^{16}} + \ldots \\ &= 3 + \sum_{n=1}^{\infty}\frac{1425}{10^{4n}} = 3 + 1425\sum_{n=1}^{\infty}\left(\frac{1}{10^4}\right)^n = 3 + 1425\sum_{n=1}^{\infty}\left(\frac{1}{10000}\right)^n \\ &= 3 + 1425\sum_{n=1}^{\infty}\frac{1}{10000}\left(\frac{1}{10000}\right)^{n-1} = 3 + 1425\frac{\frac{1}{10000}}{1-\frac{1}{10000}} \\ &= 3 + 1425\frac{1}{10000}\frac{10000}{9999} = 3 + \frac{1425}{9999} = \frac{29997+1425}{9999} = \frac{31422}{9999} = \frac{10474}{3333}\end{aligned}$$

2. A casino offers the following game to play: a fair coin is tossed until tails turns up for the first time. If this occurs on the first toss, then you receive 2 dollars; if you get heads on the first and tails on the second toss you receive $2^2 = 4$ dollars; in general, if tails turns up for the first time on the n^{th} toss you receive 2^n dollars.

Hints to set up the problem to use series: For part (a), if X is the amount of money you would pay to play this game, the expected value of X is given by

$$\mathbb{E}[X] = x_1p_1 + x_2p_2 + x_3p_3 + \ldots = \sum_{n=1}^{\infty} x_n p_n$$

where the x_n represent outcomes (how much money you win on the n^{th} toss) and p_n represents the probability of event x_n occurring. For part (d), X is the amount of games you play, x_n is represents the toss number you are on, and p_n represents the probability of getting a tails on that toss number.

Answer and explain your solutions to the following questions:

(a) What would be a fair price to pay the casino for playing the game? In other words, what is $\mathbb{E}[X]$ for our game?

(b) What does your result from part (a) mean in real life terms?

(c) What is the probability that the game stops in a finite number of tosses?

(d) What is the expected length of the game? In other words, what is the expected number of flips before you get tails?

Solution:

(a) To use the hint, we need to figure out the outcomes x_n and probabilities p_n. From the story, the outcomes are the winnings on flip n, so we have $x_n = 2^n$. What is the probability outcome x_1, flipping tails for a fair coin on the first flip? This is easy, $p_1 = \dfrac{1}{2}$. What is the probability outcome x_2, flipping heads then tails (HT)? This is just $p_2 = \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{2^2}$. What is the probability outcome x_3, flipping HHT? This is $p_3 = \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{2^3}$. We can see the pattern now that $p_n = \dfrac{1}{2^n}$. Now use the formula

$$\mathbb{E}[X] = \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty$$

(b) What does this mean? In real world terms, it means we should pay **any** amount of money to play this game. Does this make sense? Imagine paying \$20 to play, and you roll T, or HT, or HHT, or HHHT, or HHHHT. You would only win \$$2, 4, 8, 16$, and 32 respectively, but you paid \$20 to play. Would you risk your \$20 in the attempt to get 4 or more heads in a row? A rational person would enter a game if and only if the price of playing was less than $\mathbb{E}[X]$ (**This means you would only play if you expect to win more money than you pay for the game**). In this game, any price will be smaller than $\mathbb{E}[X]$. Thus, a rational person would play no matter how large the price to play is. But it seems obvious that some prices are too high for a rational person play, just consider our \$20 example. Imagine if you paid more than \$20 (especially after doing part(d))!

(c) We have essentially answered this question in part (a). We can ask the question in part (c) in a different way. We can ask, what is the probability of getting tails in a finite number of flips? If we get tails, the game is over. Let X_n be the event that tails occurs on n^{th} flip. Recall from part (a), the probability of getting all heads, and then tails on the n^{th} toss is $p_n = \dfrac{1}{2^n}$. Therefore, if we collect all of the outcomes X_n for all n, we get all possibilities of ending in finite time. So for each X_n we have its corresponding p_n. So collecting all of the X_n means we must sum all their probabilities, which means

$$P(\text{ending in finite time}) = \sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

This means with probability 1, the game will end in finite time. In other words, the game will end for some n, and you will receive 2^n dollars.

(d) To use the hint, we need to figure out the outcomes x_n and probabilities p_n. From the hint, the outcomes is the number of the flip you are on, n, so we have $x_n = n$. What is the probability for outcome x_n, flipping tails for a fair coin on the n^{th} toss? It is the same as before, $p_n = \dfrac{1}{2^n}$. Now use the formula

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{n=1}^{\infty} n\frac{1}{2^n} = 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 3\frac{1}{8} + 4\cdot\frac{1}{16} + 5\cdot\frac{1}{32} + \dots \\
&= \frac{1}{2} + \cdot\frac{(1+1)}{4} + \frac{(1+2)}{8} + \cdot\frac{(1+3)}{16} + \cdot\frac{(1+4)}{32} + \dots \\
&= \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots\right) + \left(\frac{1}{4} + \frac{2}{8} + \frac{3}{16} + \frac{4}{32} + \dots\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{2}\left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \frac{4}{16} + \dots\right) \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{2}\sum_{n=1}^{\infty} n\frac{1}{2^n} \quad \text{NOTE: the second sum is the original series!} \\
\Rightarrow \quad \mathbb{E}[X] &= \sum_{n=1}^{\infty} \frac{1}{2^n} + \frac{1}{2}\mathbb{E}[X] \\
\Rightarrow \quad \frac{1}{2}\mathbb{E}[X] &= \sum_{n=1}^{\infty} \frac{1}{2^n} \\
\Rightarrow \quad \frac{1}{2}\mathbb{E}[X] &= 1 \\
\Rightarrow \quad \mathbb{E}[X] &= 2
\end{aligned}$$

which tells us that, on average, the game will be over after 2 tosses!

This problem is called the **St. Petersburg Paradox**. What is the paradox? The paradox comes from putting together parts (a)-(d). Suppose you take out a loan of 1 trillion dollars to play this game. You would have to flip the first tails on the 41^{st} roll to make money if you play for 1 trillion dollars. The probability of this happening is $\dfrac{1}{2^{41}} \approx 4.547 \times 10^{-13}$, which is basically zero. However, there is positive probability, albeit extremely small, that you will win an insanely large amount of money. The chances of winning large amounts of money are extremely small, but the payoffs associated with these small probabilities are much greater than the probabilities are tiny. If you had unlimited credit and unlimited time, you would be able to payoff the amount you took out the loan for, and have tons of money left over. **In terms of math, we have competing levels of infinity**!

3. Use the Integral Test to determine if the following series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\arctan(n)}{1+n^2}$$

__Solution__:

Note that we have the condition $a_n = f(n)$ for $f(x) = \dfrac{\arctan(x)}{1+x^2}$. We must show that $f(x)$ is continuous, positive, and decreasing on $[1, \infty)$.

Continuity: $g(x) = \arctan(x)$ is defined for all x, therefore is continuous, and $h(x) = (1 + x^2)$ is a polynomial, so it is continuous. The quotient of continuous functions is continuous as long as the denominator is not zero, and $h(x) = 1 + x^2$ is never zero for any x. Therefore the quotient $g(x)/h(x)$ is continuous.

Positive: Since $x > 0$ on $[1, \infty)$, then $\arctan(x) > 0$ on $[1, \infty)$, and $1 + x^2 > 0$ for all x, then $f(x)$ is positive on $[1, \infty)$.

Decreasing: To show decreasing, we can show the derivative is negative:

$$f'(x) = \frac{(1+x^2)\cdot\frac{1}{1+x^2} - \arctan(x)(2x)}{(1+x^2)^2} = \frac{1 - 2x\arctan(x)}{(1+x^2)^2} < 0$$

where the inequality follows from fact that $1 - 2x\arctan(x) < 0$ for $x \geq 1$. This is because for $x = 1$, we have $\arctan(1) = \dfrac{\pi}{4} \approx 0.785$ and $2(1)(0.785) = 1.57$, so $1 - 2(1)\arctan(1) \approx 1 - 1.57 < 0$ which is the first term. For $x > 1$, the functions x and $\arctan(x)$ are both increasing, so the top will stay negative. So the numerator will always be negative for our interval.

Now we can apply the Integral Test:

$$\begin{aligned}
\int_1^{\infty} \frac{\arctan(x)}{1+x^2}\,dx &= \lim_{t\to\infty} \int_1^{t} \frac{\arctan(x)}{1+x^2}\,dx \\
&= \lim_{t\to\infty} \int_{\frac{\pi}{4}}^{\arctan(t)} u\,du \qquad \text{let } u = \arctan(x),\ du = \frac{1}{1+x^2}\,dx \\
&= \lim_{t\to\infty} \left.\frac{u^2}{2}\right|_{\frac{\pi}{4}}^{\arctan(t)} = \lim_{t\to\infty} \frac{(\arctan(t))^2}{2} - \frac{(\frac{\pi}{4})^2}{2} \\
&= \frac{\pi^2}{8} - \frac{\pi^2}{32} = \frac{3\pi^2}{32} < \infty \qquad \Rightarrow \quad \text{convergent by Integral Test}
\end{aligned}$$

Please, show all work.