Last Name: First Name:

MATH 009C - Summer 2018

Worksheet 6: July 31, 2018

1. Use the Direct Comparison Test to determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{3}{4^n+1}$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{n^3}{n^5+4}$$

<u>Solution</u>:

(a) We will compare this series with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4^n}$, which is a convergent geometric series, since $r = \frac{1}{4}$. We also have that $a_n > 0$ and $b_n > 0$ for all $n \geq 1$. All that is left is to prove the inequality for the convergent case of the Direct Comparison Test. The inequality follows from

$$\begin{aligned} 4^n &\leq 4^n + 1 \qquad \text{for } n \geq 1 \\ \frac{1}{4^n+1} &\leq \frac{1}{4^n}. \end{aligned}$$

We can also multiply the inequality by 3 to get exactly the series in the problem. We now establish that

$$\sum_{n=1}^{\infty} \frac{3}{4^n+1} \leq \sum_{n=1}^{\infty} \frac{3}{4^n} < \infty.$$

So by Direct Comparison Test, the series converges.

(b) We will compare this series with $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series. We pick this since the leading powers give us $\frac{n^3}{n^5} = \frac{1}{n^2}$. We also have that $a_n > 0$ and $b_n > 0$ for all $n \geq 1$. All that is left is to prove the inequality for the convergent case of the Direct Comparison Test. The inequality follows from

$$\begin{aligned} n^5 &\leq n^5 + 4 \qquad \text{for } n \geq 1 \\ \frac{1}{n^5+4} &\leq \frac{1}{n^5} \\ \frac{n^3}{n^5+4} &\leq \frac{n^3}{n^5} = \frac{1}{n^2} \qquad \text{since } n^3 > 0 \quad \text{for } n \geq 1 \end{aligned}$$

We now establish that

$$\sum_{n=1}^{\infty} \frac{n^3}{n^5+4} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

So by Direct Comparison Test, the series converges.

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2. Use the Limit Comparison Test to determine whether the series converges or diverges.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{n^3+3n^2+7n+1}{\sqrt{4n^{10}+4n^6+9n^2+2}}$$

$$(b) \qquad \sum_{n=1}^{\infty} \frac{5n}{n^2+|\sin(n)|}$$

$$(c) \qquad \sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$$

Solution:

(a) We should apply the limit comparison test due to the polynomials. The terms in the series are all positive, so we can apply limit comparison test. For these, take the highest powers of the numerator and the denominator, including any roots that are present to find out what to compare with

$$\frac{n^3+3n^2+7n+1}{\sqrt{4n^{10}+4n^6+9n^2+2}} \quad \Rightarrow \quad \frac{n^3}{\sqrt{4n^{10}}} = \frac{n^3}{2n^5} = \frac{1}{2n^2}.$$

So the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n^2}$ would give a convergent p-series. So we expect this series to be convergent. Take $\sum a_n$ to be our series and $\sum b_n = \sum \dfrac{1}{n^2}$. We now compute the limit and use some algebra

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\frac{n^3+3n^2+7n+1}{\sqrt{4n^{10}+4n^6+9n^2+2}}}{\frac{1}{2n^2}} = \lim_{n\to\infty} \frac{2n^5+6n^4+14n^3+2n^2}{\sqrt{4n^{10}+4n^6+9n^2+2}} \\
&= \lim_{n\to\infty} \frac{2n^5+6n^4+14n^3+2n^2}{\sqrt{4n^{10}+4n^6+9n^2+2}} \cdot \frac{\frac{1}{n^5}}{\frac{1}{\sqrt{n^{10}}}} \\
&= \lim_{n\to\infty} \frac{2+\frac{6}{n}+\frac{14}{n^2}+\frac{2}{n^3}}{\sqrt{4+\frac{4}{n^4}+\frac{9}{n^8}+\frac{2}{n^{10}}}} = \frac{2}{\sqrt{4}} = 1
\end{aligned}$$

Since the limit is a finite positive number, our series is convergent since $\sum b_n$ is convergent. **NOTE:** Just because the limit here is finite **does not imply** the series converges. Remember the theorem statement. So by limit comparison test, the series converges.

(b) This problem may seem tricky due to the $|\sin(n)|$ term. The terms in the series are all positive, so we can apply limit comparison test. For these, take the highest powers of the numerator and the denominator, including any roots that are present to find out what to compare with

$$\frac{5n}{n^2+|\sin(n)|} \quad \Rightarrow \quad \frac{n}{n^2} = \frac{1}{n}.$$

So the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ would give a divergent p-series. So we expect this series to be divergent. Take $\sum a_n$ to be our series and $\sum b_n = \sum \dfrac{1}{n}$. We now compute the limit and use some algebra

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{5n}{n^2+|\sin(n)|}}{\frac{1}{n}} &= \lim_{n\to\infty} \frac{5n^2}{n^2+|\sin(n)|} \\
&= \lim_{n\to\infty} \frac{5n^2}{n^2+|\sin(n)|} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} \\
&= \lim_{n\to\infty} \frac{5}{1+\frac{|\sin(n)|}{n^2}} = \frac{5}{1} = 5
\end{aligned}$$

Formally, we show the term with sine goes to zero by squeeze theorem

$$\begin{aligned}
0 \le \frac{|\sin(n)|}{n^2} &\le \frac{1}{n^2} \\
\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{|\sin(n)|}{n^2} &\le \lim_{n\to\infty} \frac{1}{n^2} \\
0 \le \lim_{n\to\infty} \frac{|\sin(n)|}{n^2} &\le 0 \\
\Rightarrow \quad \lim_{n\to\infty} \frac{|\sin(n)|}{n^2} &= 0
\end{aligned}$$

Since the limit is a finite positive number, our series is divergent since $\sum b_n$ is divergent. So by limit comparison test, the series diverges.

(c) This problem is like the homework question, except the choice of comparison is slightly different. The terms in the series are all positive, so we can apply limit comparison test. For this problem, we take $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series. So we expect this series to be convergent. Take $\sum a_n$ to be our series and $\sum b_n = \sum \frac{1}{n^2}$. We now compute the limit and use a substitution $u = \frac{1}{n^2}$ to get

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{u\to 0^+} \frac{\sin(u)}{u} = 1$$

or use L'Hopital's rule. Since the limit is a finite positive number, our series is convergent since $\sum b_n$ is convergent. So by limit comparison test, the series converges.

3. Use the Ratio or Root test to determine if the series is convergent or divergent.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$$

$$(b) \qquad \sum_{n=1}^{\infty} (\arctan(n))^n$$

Solution:

(a) Use the Ratio Test

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(2(n+1))!}{((n+1)!)^2} \cdot \frac{(n!)^2}{(2n)!}\right| \\
&= \lim_{n\to\infty} \frac{(2(n+1))!}{(2n)!} \cdot \frac{(n!)^2}{((n+1)!)^2} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)((2n)!)}{(2n)!} \cdot \frac{(n!)(n!)}{((n+1)!)((n+1)!)} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)((2n)!)}{(2n)!} \cdot \frac{(n!)(n!)}{(n+1)(n!)(n+1)(n!)} \\
&= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \\
&= \lim_{n\to\infty} \frac{4n^2+6n+2}{n^2+2n+1} = 4 > 1
\end{aligned}$$

Since the value of the limit is greater than 1, the series diverges by the Ratio Test.

(b) Use the Root Test

$$\begin{aligned}
\lim_{n\to\infty} |a_n|^{\frac{1}{n}} &= \lim_{n\to\infty} |(\arctan(n))^n|^{\frac{1}{n}} \\
&= \lim_{n\to\infty} ((\arctan(n))^n)^{\frac{1}{n}} \\
&= \lim_{n\to\infty} \arctan(n) = \frac{\pi}{2} > 1
\end{aligned}$$

Since the value of the limit is greater than 1, the series diverges by the Root Test.

Please, show all work.