Last Name: First Name:

MATH 009C - Summer 2018

Worksheet 7: August 7, 2018

1. Determine the radius and interval of convergence for the following power series.

$$(a) \qquad \sum_{n=1}^{\infty} \frac{n^3}{3^n}(x+1)^n$$

$$(b) \qquad \sum_{n=1}^{\infty} (-1)^n \frac{2^n}{n!}(x-2)^n$$

Solution:

(a) Use the Ratio Test to find the radius of convergence

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(n+1)^3(x+1)^{n+1}}{3^{n+1}} \cdot \frac{3^n}{n^3(x+1)^n}\right| \\
&= \lim_{n\to\infty}\frac{3^n}{3^{n+1}} \cdot \frac{(n+1)^3}{n^3} \cdot |x+1| \\
&= \frac{1}{3}|x+1| \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^3 \\
&= \frac{1}{3}|x+1| < 1 \quad \text{for convergence by Ratio Test} \\
&\Rightarrow \quad |x+1| < 3 = R
\end{aligned}$$

so the radius of convergence is $R = 3$. Solve the inequality for the interval of convergence, to **tentatively** $(-4, 2)$. Now check the endpoints

$$\text{for } x=-4 \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{n^3}{3^n}(-3)^n = \sum_{n=1}^{\infty} n^3(-1)^n \quad \Rightarrow \quad \text{divergent by Divergence Test}$$

$$\text{for } x=2 \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{n^3}{3^n}(3)^n = \sum_{n=1}^{\infty} n^3 \quad \Rightarrow \quad \text{divergent by Divergence Test}$$

Therefore the interval of convergence is $(-4, 2)$.

(b) Use the Ratio Test to find the radius of convergence

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}2^{n+1}(x-2)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n 2^n (x-2)^n}\right| \\
&= \lim_{n\to\infty}\frac{2^{n+1}}{2^n} \cdot \frac{n!}{(n+1)n!} \cdot |x-2| \\
&= 2|x-2| \lim_{n\to\infty}\frac{1}{n+1} \\
&= 0 < 1 \quad \text{for convergence by Ratio Test}
\end{aligned}$$

which is true for all x. So the radius of convergence is $R = \infty$, and the interval of convergence is $(-\infty, \infty)$.

Please, show all work.

2. Use the definition of Taylor series to compute the 3rd order Taylor polynomial $T_3(x)$ for the following function centered at $a = 0$. **NOTE: Do not use the substitution question to do this problem, or you will receive no credit**.

$$f(x) = e^{-3x}$$

<u>**Solution**</u>**:**

To use the definition of Taylor series, we use the formula on the front of the exam

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

For our question, $f(x) = e^{-3x}$, $a = 0$, and we are computing $T_3(x)$. Recall that this means there are 4 terms to compute. We need to compute the coefficients which are generated using the derivatives of $f(x)$. So we get

$$\begin{aligned} f^{(0)}(x) &= e^{-3x} \quad \Rightarrow \quad f^{(0)}(0) = 1 \\ f^{(1)}(x) &= -3e^{-3x} \quad \Rightarrow \quad f^{(1)}(0) = -3 \\ f^{(2)}(x) &= 9e^{-3x} \quad \Rightarrow \quad f^{(2)}(0) = 9 \\ f^{(3)}(x) &= -27e^{-3x} \quad \Rightarrow \quad f^{(3)}(0) = -27 \end{aligned}$$

Now we can compute the whole coefficient by dividing by the $n!$ to get

$$\begin{aligned} \frac{f^{(0)}(0)}{0!} &= 1 \\ \frac{f^{(1)}(0)}{1!} &= -3 \\ \frac{f^{(2)}(0)}{2!} &= \frac{9}{2} \\ \frac{f^{(3)}(0)}{3!} &= -\frac{27}{6} = -\frac{9}{2} \end{aligned}$$

Therefore, the Taylor series approximation of $f(x)$ is

$$f(x) = e^{-3x} \approx T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(a)}{n!}(x-a)^n = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3$$

Please, show all work.

3. Determine the Taylor Series for the following functions.

$$(a) \qquad \int \cos(x^3)\ dx$$

$$(b) \qquad \frac{x - \arctan(x)}{x^2}$$

Solution:

(a) We know from the table on the front (line 1 below)

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

$$\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n}}{(2n)!}$$

$$\Rightarrow \quad \int \cos(x^3)\ dx = \int \left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!} \right)\ dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\int x^{6n}\ dx \right)$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{x^{6n+1}}{6n+1}$$

(b) Use the formula for the Taylor Series centered at 0 from the front page of the test:

$$\arctan(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Compute the numerator first by writing out the terms for the above series

$$x - \arctan(x) = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \dots$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{2n+1}.$$

Now multiplying the above series by $\frac{1}{x^2}$, we get the desired Taylor Series

$$\frac{x - \arctan(x)}{x^2} = \frac{1}{x^2} \sum_{n=1}^{\infty}(-1)^{n+1} \frac{x^{2n+1}}{2n+1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1} \frac{x^{2n-1}}{2n+1}.$$

Please, show all work.