MATH 010B - Spring 2018

Worked Problems - Section 5.1

1. Compute the following iterated integral

$$\int_1^2 \int_2^4 \left(\frac{x}{y} + \frac{y}{x}\right)\ dx\ dy$$

Solution: As the integrand is symmetric in x and y (note the terms are reciprocals of each other), we can integrate in the order given, as switching the order won't simplify anything:

$$\begin{aligned}
\int_1^2 \int_2^4 \left(\frac{x}{y} + \frac{y}{x}\right)\ dx\ dy &= \int_1^2 \left(\frac{x^2}{2y} + y\ln|x|\right)\Big|_{x=2}^{x=4}\ dy \\
&= \int_1^2 \left(\frac{8}{y} + y\ln(4) - \left(\frac{2}{y} + y\ln(2)\right)\right)\ dy \\
&= \int_1^2 \left(\frac{6}{y} + (\ln(4) - \ln(2))y\right)\ dy \\
&= \left(6\ln|y| + \frac{\ln(2)}{2}y^2\right)\Big|_1^2 \\
&= 6\ln(2) + 2\ln(2) - \frac{1}{2}\ln(2) \\
&= \frac{15}{2}\ln(2)
\end{aligned}$$

□

2. Compute the following iterated integral

$$\int_0^1 \int_0^1 \left(8xye^{x+y}\right)\ dx\ dy$$

Solution: By direct computation (note that in the second line, we can write the integrand as $f(x)g(y)$, so we use the product trick. If you don't see the trick, you can do this the same way as the problem above.)

$$\begin{aligned}
\int_0^1 \int_0^1 \left(8xye^{x+y}\right)\ dx\ dy &= \int_0^1 \int_0^1 \left(8xye^x e^y\right)\ dx\ dy \\
&= \left(\int_0^1 8xe^x\ dx\right) \cdot \left(\int_0^1 ye^y\ dy\right) \\
&= \left(8e^x(x-1)|_0^1\right) \cdot \left(e^y(y-1)|_0^1\right) \\
&= (8)(1) = 8
\end{aligned}$$

□

3. Compute the following iterated integral

$$\int_{-1}^{0}\int_{1}^{2} (-x\ln(y))\ dy\ dx$$

***Solution*:** By direct computation (note that in the first line, we can write the integrand as $f(x)g(y)$, so we use the product trick. If you don't see the trick, you can do this the same way as the problem 1.)

$$\begin{aligned}
\int_{-1}^{0}\int_{1}^{2} (-x\ln(y))\ dy\ dx &= \left(\int_{-1}^{0} -x\ dx\right)\cdot\left(\int_{1}^{2} \ln(y)\ dy\right)\\
&= \left(-\frac{x^2}{2}\Big|_{-1}^{0}\right)\cdot\left(y\ln(y) - y|_1^2\right)\\
&= \left(\frac{1}{2}\right)(2\ln(2) - 2 - (1\ln(1) - 1))\\
&= \ln(2) - \frac{1}{2}
\end{aligned}$$

□

4. Compute the following iterated integral

$$\int_{0}^{1}\int_{0}^{1} (5x+2y)^7\ dx\ dy$$

***Solution*:** By direct computation

$$\begin{aligned}
\int_{0}^{1}\int_{0}^{1} (5x+2y)^7\ dx\ dy &= \frac{1}{5}\int_{0}^{1}\int_{2y}^{5+2y} u^7\ du\ dy \qquad u = 5x+2y,\ du = 5\ dx\\
&= \frac{1}{5}\int_{0}^{1} \frac{u^8}{8}\Big|_{u=2y}^{u=5+2y}\ dy\\
&= \frac{1}{5}\int_{0}^{1}\left(\frac{(5+2y)^8}{8} - \frac{(2y)^8}{8}\right)\ dy\\
&= \frac{1}{5}\int_{0}^{1}\left(\frac{(5+2y)^8}{8} - \frac{2^8y^8}{8}\right)\ dy\\
&= \frac{1}{40}\int_{0}^{1}(5+2y)^8\ dy - \frac{2^8}{40}\int_{0}^{1} y^8\ dy\\
&= \frac{1}{80}\int_{5}^{7} u^8\ dy - \frac{2^8}{40}\int_{0}^{1} y^8\ dy \qquad u = 5+2y,\ du = 2\ dy\\
&= \frac{1}{80}\left(\frac{u^9}{9}\Big|_5^7\right) - \frac{2^8}{40}\left(\frac{y^9}{9}\Big|_0^1\right)\\
&= \frac{1}{80}\left(\frac{7^9}{9} - \frac{5^9}{9}\right) - \frac{32}{5}\left(\frac{1}{9}\right)\\
&= \frac{1}{80}\frac{38400482}{9} - \frac{32}{45}\\
&= \frac{19200241}{360} - \frac{32}{45}\\
&= \frac{1279999}{24}
\end{aligned}$$

□

5. Find the volume bounded by the graph of $f(x,y) = x^4 + y^2$, the rectangle $R = [-1,1] \times [-7,-2]$, and the four vertical sides of the rectangle R.

***Solution*:** Note that the region being described is just a rectangular prism, where the base is the rectangle R, and the top of the prism is the surface $f(x,y) = x^4 + y^2$. Now compute the volume

$$\begin{aligned}
\iint_R f(x,y)\, dA &= \int_{-1}^{1} \int_{-7}^{-2} \left(x^4 + y^2\right)\, dy\, dx \\
&= \int_{-1}^{1} \left(yx^4 + \frac{y^3}{3} \Big|_{y=-7}^{y=-2} \right)\, dx \\
&= \int_{-1}^{1} \left(-2x^4 - \frac{8}{3} + 7x^4 + \frac{7^3}{3} \right)\, dx \\
&= \int_{-1}^{1} \left(5x^4 + \frac{7^3 - 8}{3} \right)\, dx \\
&= \left(x^5 + \frac{7^3 - 8}{3} x \Big|_{-1}^{1} \right) \\
&= 1 + \frac{7^3 - 8}{3} - \left(-1 - \frac{7^3 - 8}{3} \right) \\
&= 2 + \frac{2}{3}(7^3 - 8) \\
&= \frac{676}{3}
\end{aligned}$$

□