## MATH 010B - Spring 2018

Worked Problems - Section 5.4

**1.** Evaluate the integral.

$$\int_0^{\pi/2}\int_0^{\cos(\theta)} 9\cos(\theta)\ dr\ d\theta$$

***Solution*:** The bounds here are $0 \le \theta \le \frac{\pi}{2}$, and $0 \le r \le \cos(\theta)$. Note that $r = \cos(\theta)$ is the graph of a circle with center $(0.5, 0)$, with radius $\frac{1}{2}$, but since the values of $\theta$ are restricted we only have the top half of the circle. We compute

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{\cos(\theta)} 9\cos(\theta)\ dr\ d\theta &= \int_0^{\pi/2} 9r\cos(\theta)\Big|_0^{\cos(\theta)}\ d\theta \\
&= 9\int_0^{\pi/2} \cos^2(\theta)\ d\theta \\
&= \frac{9}{2}\int_0^{\pi/2} 1+\cos(2\theta)\ d\theta \\
&= \frac{9}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\Big|_0^{\pi/2}\right) \\
&= \frac{9\pi}{4}
\end{aligned}$$

To do this integral by changing the bounds, we would have the top half circle region bounded by $0 \le r \le 1$, and $0 \le \theta \le \cos^{-1}(r)$. We then compute

$$\begin{aligned}
\int_0^1\int_0^{\cos^{-1}(r)} 9\cos(\theta)\ d\theta\ dr &= \int_0^1 9\sin(\theta)\Big|_0^{\cos^{-1}(r)}\ dr \\
&= 9\int_0^1 \sin(\cos^{-1}(r))\ dr \\
&= 9\int_0^1 \sqrt{1-r^2}\ dr \\
&= 9\int_0^{\sin(1)} \sqrt{1-\sin^2(u)}\cos(u)\ du \quad \text{use } r=\sin(u),\ dr=\cos(u)\ du \\
&= 9\int_0^{\sin(1)} \cos^2(u)\ du = \frac{9}{2}\int_0^{\sin(1)} 1+\cos(2u)\ du \\
&= \frac{9}{2}\left(u+\frac{1}{2}\sin(2u)\Big|_0^{\sin(1)}\right) \quad \text{use } \sin(2u)=2\sin(u)\cos(u) \\
&= \frac{9}{2}\left(\sin^{-1}(r)+r\sqrt{1-r^2}\Big|_0^1\right) = \frac{9}{2}(\sin^{-1}(1)) = \frac{9}{2}\cdot\frac{\pi}{2} \\
&= \frac{9\pi}{4}
\end{aligned}$$

where we have made a special substitution. Note that for $0 \le r \le 1$, and $\theta = \cos^{-1}(r)$, then $0 \le \theta \le \frac{\pi}{2}$, which is what we had originally. Now, $\sin(\theta) \ge 0$ for $0 \le \theta \le \frac{\pi}{2}$, so we take the positive root below, and we have that

$$\sin(\cos^{-1}(r)) = \sin(\theta) = \sqrt{1-\cos^2(\theta)} = \sqrt{1-r^2}$$

since we have the relation $\theta = \cos^{-1}(r) \iff r = \cos(\theta)$. $\square$

**2.** Evaluate the integral.

$$\int_0^{20}\int_{y/4}^{5} e^{x^2}\ dx\ dy$$

***Solution*:** Note that we are integrating over the triangular region bounded by the $x$-axis, $x = 5$, and the line $x = \frac{y}{4} \iff y = 4x$. We can rewrite the integral by using the bounds $0 \le x \le 5$ and $0 \le y \le 4x$, which makes the integration easier

$$\begin{aligned}
\int_0^{20}\int_{y/4}^{5} e^{x^2}\ dx\ dy &= \int_0^{5}\int_{0}^{4x} e^{x^2}\ dy\ dx \\
&= \int_0^5 ye^{x^2}\Big|_{y=0}^{y=4x}\ dx \\
&= \int_0^5 4xe^{x^2}\ dx \quad \text{use } u = x^2,\ du = 2x\ dx \\
&= 2\int_0^{25} e^u\ du \\
&= 2(e^{25} - 1)
\end{aligned}$$

□

**3.** Change the order of integration and evaluate.

$$\int_0^{1}\int_{\sqrt{y}}^{1} e^{3x^3}\ dx\ dy$$

***Solution*:** Note that we are integrating over the region bounded by the $x$-axis, $x = 1$, and the parabola $x = \sqrt{y} \iff y = x^2$. We can rewrite the integral by using the bounds $0 \le x \le 1$ and $0 \le y \le x^2$, which makes the integration easier

$$\begin{aligned}
\int_0^{1}\int_{\sqrt{y}}^{1} e^{3x^3}\ dx\ dy &= \int_0^{1}\int_{0}^{x^2} e^{3x^3}\ dy\ dx \\
&= \int_0^1 ye^{3x^3}\Big|_{y=0}^{y=x^2}\ dx \\
&= \int_0^1 x^2e^{3x^3}\ dx \quad \text{use } u = 3x^3,\ du = 9x^2\ dx \\
&= \frac{1}{9}\int_0^{3} e^u\ du \\
&= \frac{1}{9}\left(e^3 - 1\right)
\end{aligned}$$

□