MATH 010B - Spring 2018

Worked Problems - Section 5.5

1. Evaluate the integral.

$$\iiint_R 2x^2 \; dx \; dy \; dz$$

where $R = [0,2] \times [0,2] \times [0,2]$.

***Solution*:** Note that the region R is just a cube in the first octant with sides of length 2. Therefore, we have

$$\begin{aligned}
\iiint_R = 2x^2 \; dx \; dy \; dz &= \int_0^2 \int_0^2 \int_0^2 2x^2 \; dx \; dy \; dz \\
&= \left(\int_0^2 2x^2 \; dx \right) \left(\int_0^2 \; dy \right) \left(\int_0^2 \; dz \right) \\
&= \left(\frac{2}{3} x^3 \bigg|_0^2 \right) (2) \, (2) \\
&= \frac{64}{3}
\end{aligned}$$

□

2. Evaluate the integral.

$$\iiint_R = (3x + 6y + z) \; dx \; dy \; dz$$

where $R = [0,3] \times [-1,1] \times [0,2]$.

***Solution*:** We compute the integral given the bounds above

$$\begin{aligned}
\iiint_R = (3x + 6y + z) \; dx \; dy \; dz &= \int_0^2 \int_{-1}^1 \int_0^3 (3x + 6y + z) \; dx \; dy \; dz \\
&= \int_0^2 \int_{-1}^1 \left(\frac{3}{2} x^2 + 6yx + zx \right) \bigg|_{x=0}^{x=3} \; dy \; dz \\
&= \int_0^2 \int_{-1}^1 \left(\frac{27}{2} + 18y + 3z \right) \; dy \; dz \\
&= \int_0^2 \left(\frac{27}{2} y + 9y^2 + 3yz \right) \bigg|_{y=-1}^{y=1} \; dz \\
&= \int_0^2 (27 + 6z) \; dz \\
&= \left(27z + 3z^2 \right) \big|_0^2 \\
&= 66
\end{aligned}$$

□

3. Find the volume of the region bounded by $z = x^2 + y^2$ and $z = 10 - x^2 - 2y^2$.

***Solution*:** The graph of $z = x^2 + y^2$ is a (circular cross-section) paraboloid that opens upward, and $z = 10 - x^2 - 2y^2$ is an elliptic paraboloid that opens downward. Note that in the first line below we can represent the volume as a triple integral, or equivalently as a double integral. As a double integral, we must figure out what the region R is. The region R is the projection into the xy-plane, which can be determined by:

$$\begin{gathered} x^2 + y^2 = 10 - x^2 - 2y^2 \\ \frac{x^2}{5} + \frac{3y^2}{10} = 1 \end{gathered}$$

which gives us an equation for an ellipse. Therefore, we have the bounds $-\sqrt{5} \leq x \leq \sqrt{5}$ and $-\sqrt{\frac{10}{3} - \frac{2}{3}x^2} \leq y \leq \sqrt{\frac{10}{3} - \frac{2}{3}x^2}$ for the region R. As you can see below, this does not look like a very nice integral to compute. If use a modified version of polar coordinates: $x = \sqrt{5} r\cos(\theta)$ and $y = \frac{10}{3} r\sin(\theta)$, then we are mapping the ellipse to the unit disk (the circle centered at the origin, with radius 1 and its interior). Therefore, $dA = r\sqrt{\frac{50}{3}}\ dr\ d\theta$ by computing the Jacobian of our substitution above. Thus we only need to find the bounds for the unit disk, which are easy: $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$.

$$\begin{aligned} \iiint_W dV &= \iint_R \int_{x^2+y^2}^{10-x^2-2y^2} dz\ dA = \iint_R (10 - 2x^2 - 3y^2)\ dA \\ &= \int_{-\sqrt{5}}^{\sqrt{5}} \int_{-\sqrt{\frac{10}{3}-\frac{2}{3}x^2}}^{\sqrt{\frac{10}{3}-\frac{2}{3}x^2}} (10 - 2x^2 - 3y^2)\ dy\ dx \\ &= \int_0^{2\pi} \int_0^1 \left(10 - 2(5r^2\cos^2(\theta)) - 3\left(\frac{10}{3}r^2\sin^2(\theta)\right)\right) r\sqrt{\frac{50}{3}}\ dr\ d\theta \\ &= 10\sqrt{\frac{50}{3}} \int_0^{2\pi} \int_0^1 (r - r^3)\ dr\ d\theta \\ &= 20\pi\sqrt{\frac{50}{3}} \left(\frac{r^2}{2} - \frac{r^4}{4}\Big|_0^1\right) \\ &= 100\pi\sqrt{\frac{2}{3}} \left(\frac{1}{2} - \frac{1}{4}\right) \\ &= 25\pi\sqrt{\frac{2}{3}} \end{aligned}$$

□

4. Evaluate the integral.

$$\iiint_W z \; dx \; dy \; dz$$

where W is the region bounded by the planes $x = 0, y = 0, z = 0, z = 1$, and the cylinder $x^2 + y^2 = 9$, with $x \geq 0$ and $y \geq 0$.

***Solution*:** Note that we are integrating over a quarter of the cylinder, the part that is in the first octant. The first octant is determined as we have the restriction by the planes $x = 0, y = 0, z = 0$. The cylinder goes from a base at $z = 0$ to the top at $z = 1$. The cylinder has radius 3, as the cross sections are given by the circles $x^2 + y^2 = 9$. Therefore, the bounds are $0 \leq z \leq 1$, $0 \leq y \leq 3$, and $0 \leq x \leq \sqrt{9 - y^2}$. So the integral becomes

$$\begin{aligned}
\iiint_W z \; dx \; dy \; dz &= \int_0^1 \int_0^3 \int_0^{\sqrt{9-y^2}} z \; dx \; dy \; dz \\
&= \int_0^1 z \left(\int_0^3 \int_0^{\sqrt{9-y^2}} dx \; dy \right) \; dz \\
&= \frac{9\pi}{4} \int_0^1 z \; dz = \frac{9\pi}{4} \left(\left. \frac{z^2}{2} \right|_0^1 \right) = \frac{9\pi}{8}
\end{aligned}$$

where we have used the fact that the double integral in parenthesis in the second line is just the area of $\frac{1}{4}$ of the circle $x^2 + y^2 = 9$. The whole circle has area $A = \pi r^2 = 9\pi$, and one quarter of that area is $\frac{9\pi}{4}$. This saves us from doing the integration. $\square$

5. Evaluate the integral

$$\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz \; dy \; dx$$

***Solution*:** We compute the integral directly

$$\begin{aligned}
\int_0^1 \int_0^{2x} \int_{x^2+y^2}^{x+y} dz \; dy \; dx &= \int_0^1 \int_0^{2x} \left(x + y - x^2 - y^2 \right) \; dy \; dx \\
&= \int_0^1 \left. \left(xy + \frac{y^2}{2} - x^2 y - \frac{y^3}{3} \right) \right|_{y=0}^{y=2x} \; dx \\
&= \int_0^1 \left(2x^2 + \frac{(2x)^2}{2} - 2x^3 - \frac{(2x)^3}{3} \right) \; dx \\
&= \int_0^1 \left(4x^2 - 2x^3 - \frac{8x^3}{3} \right) \; dx \\
&= \left. \frac{4}{3}x^3 - \frac{1}{2}x^4 - \frac{2x^4}{3} \right|_0^1 \\
&= \frac{1}{6}
\end{aligned}$$

$\square$

6. Evaluate the integral

$$\iiint_W (x^2 + y^2)\ dx\ dy\ dz$$

where W is the pyramid with the top vertex at $(0, 0, 1)$ and the base vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1,$
and $(1, 1, 0)$.

***Solution*:** Note that the horizontal cross-sections of a pyramid with square base are squares. When we are at the height z, the square has one corner at $(0, 0, z)$ and side length $1 - z$. So then, we have the bounds $0 \leq x \leq 1 - z$ and $0 \leq y \leq 1 - z$. Now we can compute the integral

$$\begin{aligned}
\iiint_W (x^2 + y^2)\ dx\ dy\ dz &= \int_0^1 \int_0^{1-z} \int_0^{1-z} (x^2 + y^2)\ dx\ dy\ dz \\
&= \int_0^1 \int_0^{1-z} \left(\frac{(1-z)^3}{3} + (1-z)y^2 \right)\ dy\ dz \\
&= \int_0^1 \left(\frac{(1-z)^3}{3} y + \frac{(1-z)y^3}{3} \right) \Bigg|_0^{1-z}\ dz \\
&= \int_0^1 \left(\frac{(1-z)^4}{3} + \frac{(1-z)^4}{3} \right)\ dz \\
&= \frac{2}{3} \int_0^1 (1-z)^4\ dz \\
&= -\frac{2}{15} (1-z)^5 \Bigg|_0^1 \\
&= \frac{2}{15}
\end{aligned}$$

$\square$