MATH 010B - Spring 2018

Worked Problems - Section 6.1

1. Determine if the following functions $T : \mathbb{R}^2 \to \mathbb{R}^2$ are one-to-one and/or onto

$$\begin{aligned}
(a) &\quad T(x,y) = (8x, y)\\
(b) &\quad T(x,y) = \left(x^2, y\right)\\
(c) &\quad T(x,y) = \left(\sqrt[3]{x}, \sqrt[3]{y}\right)\\
(d) &\quad T(x,y) = (\sin(x), \cos(y))
\end{aligned}$$

***Solution*:** (a) To show one-to-one, we have the equations

$$\begin{aligned}
(8x, y) &= (8x', y')\\
\Rightarrow \quad 8x = 8x', &\qquad y = y'\\
\Rightarrow \quad x = x', &\qquad y = y'
\end{aligned}$$

therefore, by definition of one-to-one, the function $T(x,y)$ is one-to-one on all of $\mathbb{R}^2$. For onto, if we can solve the equation $T(u,v) = (x,y)$ uniquely for u and v in terms of x and y. This is equivalent to

$$\begin{aligned}
T(u,v) = (8u, v) &= (x,y)\\
\Rightarrow \quad 8u = x, &\qquad v = y\\
\Rightarrow \quad u = \frac{x}{8}, &\qquad v = y
\end{aligned}$$

Since we solved the problem uniquely, the function $T(x,y)$ is onto.

(b) For this map, we can find a counterexample to the one-to-one definition:

$$\begin{aligned}
T(x,y) &= (x^2, y)\\
T(1,1) &= (1^2, 1) = (1,1)\\
T(-1,1) &= ((-1)^2, 1) = (1,1)
\end{aligned}$$

but $(-1,1) \neq (1,1)$. Therefore by definition of one-to-one, the function $T(x,y)$ is not one-to-one on all of $\mathbb{R}^2$. For onto, if we can solve the equation $T(u,v) = (x,y)$ uniquely for u and v in terms of x and y. This is equivalent to

$$\begin{aligned}
T(u,v) = (u^2, v) &= (x,y)\\
\Rightarrow \quad u^2 = x, &\qquad v = y\\
\Rightarrow \quad u = \pm x, &\qquad v = y
\end{aligned}$$

Since the solution is not unique, the function $T(x,y)$ is not onto.

(c) To show one-to-one, we have the equations

$$\begin{aligned}
\left(\sqrt[3]{x}, \sqrt[3]{y}\right) &= \left(\sqrt[3]{x'}, \sqrt[3]{y'}\right)\\
\Rightarrow \quad \sqrt[3]{x} = \sqrt[3]{x'}, &\qquad \sqrt[3]{y} = \sqrt[3]{y'}\\
\Rightarrow \quad x = x', &\qquad y = y'
\end{aligned}$$

where we have cubed both sides of the equation. This is valid since cubed root is defined for both positive and negative values of x and y over $\mathbb{R}$ and the solution is unique. Therefore, by definition of one-to-one, the function $T(x, y)$ is one-to-one on all of $\mathbb{R}^2$. For onto, if we can solve the equation $T(u, v) = (x, y)$ uniquely for u and v in terms of x and y. This is equivalent to

$$\begin{aligned} & T(u,v) = \left(\sqrt[3]{u}, \sqrt[3]{v}\right) = (x,y) \\ \Rightarrow \quad & \sqrt[3]{u} = x, \qquad \sqrt[3]{v} = y \\ \Rightarrow \quad & u = x^3, \qquad v = y^3 \end{aligned}$$

Since the solution is unique, the function $T(x, y)$ is not onto.

(d) For this map, we can find a counterexample to the one-to-one definition:

$$\begin{aligned} T(x,y) &= (\sin(x), \cos(y)) \\ T(0,0) &= (\sin(0), \cos(0)) = (0,1) \\ T(2\pi, 2\pi) &= (\sin(2\pi), \cos(2\pi)) = (0,1) \end{aligned}$$

but $(0, 0) \neq (2\pi, 2\pi)$. Therefore by definition of one-to-one, the function $T(x, y)$ is not one-to-one on all of $\mathbb{R}^2$. For onto, if we can solve the equation $T(u, v) = (x, y)$ uniquely for u and v in terms of x and y. This is equivalent to

$$\begin{aligned} & T(u,v) = (\sin(u), \cos(v)) = (x,y) \\ \Rightarrow \quad & \sin(u) = x, \qquad \cos(v) = y \end{aligned}$$

But the solution cannot be unique on $\mathbb{R}$, because sine and cosine are periodic. Since the solution is not unique, the function $T(x, y)$ is not onto.

□

2. Let D be a parallelogram with vertices $(0,0),(-1,6),(-2,0),(-1,-6)$ and $D^* = [0,1] \times [0,1]$. Find a linear map T such that $T(D^*) = D$.

Solution: Finding the linear map T is equivalent to finding a matrix A that maps the vertices of the square to the parallelogram. Essentially we want to find the elements of the matrix A such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x^* \\ y^* \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

where (x^*, y^*) are points in the space with the square, and (x, y) are points in the space with the parallelogram. We have four conditions to satisfy since we need to map $(0,0)$ to $(0,0)$, $(1,1)$ to $(-2,0)$,

Using the matrix, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

which tells us that $b = -1$ and $d = 6$. We can do the same thing with $(x^*, y^*) = (1,0)$ and $(x,y) = (-1,-6)$, and then we deduce that $a = -1$ and $c = -6$. We can check the other two coordinates and see they also work with this choice of A. Therefore, we have the matrix A being to defined as

$$A = \begin{bmatrix} -1 & -1 \\ -6 & 6 \end{bmatrix}$$

□

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the spherical coordinate mapping defined by

$$x = \rho \sin(\varphi) \cos(\theta),$$
$$y = \rho \sin(\varphi) \sin(\theta),$$
$$z = \rho \cos(\varphi)$$

Let D^* be the set of points (ρ, φ, θ) such that $\rho \in [0,2], \varphi \in [0,\pi], \theta \in [0,2\pi]$. Answer the following questions about the image $D = T(D^*)$.

(a) D is the set of all points (x, y, z), where

- $x^2 + y^2 + z^2 \leq 2$
- $x^2 + y^2 + z^2 \geq 2$
- neither

(b) Which of the following is true about the spherical mapping $T : D^* \to D$, where for D^* is the set of points (ρ, φ, θ)?

- It is one-to-one for $\rho \in [0,2], \varphi \in [0,\pi], \theta \in [0,2\pi]$.
- It is one-to-one for $\rho \in [0,2), \varphi \in [0,\pi), \theta \in [0,2\pi]$.
- It is one-to-one for $\rho \in (0,2], \varphi \in (0,\pi], \theta \in (0,2\pi]$.
- It is one-to-one for $\rho \in (0,2], \varphi \in (0,\pi), \theta \in (0,2\pi)$.
- It is one-to-one for $\rho \in [0,2), \varphi \in [0,\pi), \theta \in [0,2\pi)$.
- none of these

***Solution*:** For (a), since we have $\varphi \in [0, \pi], \theta \in [0, 2\pi]$ we are tracing one angle in a complete circle, and the other angle in a half circle, which traces out the surface of a sphere. Now we have the radius $\rho \in [0, 2]$, which means the spheres are concentric from the origin out to the sphere with radius 2. The equation of the sphere is $x^2+y^2+z^2 = r^2$, so we care about being inside the sphere, thus the correct choice is $x^2 + y^2 + z^2 \leq 2$.

For (b), the first choice will fail due to the periodicity of the sine and cosine functions, as in problem 1(d) above. The second and fifth choice will fail due to the fact that for $\rho = 0$, we can choose any φ and θ and get the point $(x, y, z) = (0, 0, 0)$. The third condition fails since if we choose $\varphi = \pi$ and $\rho = \rho_0$, where ρ_0 is any value in $(0, 2]$, we have $(x, y, z) = (0, 0, \rho_0)$, so you can choose any θ to get the same coordinate. For the fourth choice, we encounter none of these issues, as $\rho = 0$ is not included, so the $(0, 0, 0)$ situation is eliminated, and we don't have the situation where periodicity causes the one-to-one definition to fail, as the endpoints of the intervals are not included. Alternatively, you can prove this by definition. So the solution is only choice 4. □