MATH 010B - Spring 2018

Worked Problems - Section 6.2

1. Compute the following double integral

$$\iiint_{\substack{x^2+y^2\leq 9\\ 2\leq z\leq 3}} ze^{x^2+y^2}\ dV$$

***Solution*:** Here, we can't hope to integrate this directly in Cartesian coordinates, since the the exponential function poses problems. Therefore, we will switch to cylindrical coordinates, as the region described is a cylinder. For the bounds given in terms of x, y, and z, we convert everything to cylindrical coordinates as the following:

$$\begin{aligned} dV &= r\ drd\theta dz \\ 0 &\leq r \leq 3 \\ 0 &\leq \theta \leq 2\pi \\ 2 &\leq z \leq 3 \end{aligned}$$

From here we can set up the integral. We then get

$$\begin{aligned} \iiint_{\substack{x^2+y^2\leq 9\\ 2\leq z\leq 3}} ze^{x^2+y^2}\ dV &= \int_2^3 \int_0^{2\pi} \int_0^3 ze^{r^2} r\ drd\theta dz \\ &= \left(\int_2^3 z\ dz\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^3 re^{r^2}\ dr\right) \\ &= \left(\frac{z^2}{2}\bigg|_2^3\right)(2\pi)\left(\frac{1}{2}e^{r^2}\bigg|_0^3\right) \\ &= \frac{5}{2}\pi(e^9-1) \end{aligned}$$

□

2. Compute the following double integral

$$\iiint_W e^{\sqrt{(x^2+y^2+z^2)^3}}\ dV$$

where $W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$.

***Solution*:** Here, we can't hope to integrate this directly in Cartesian coordinates, since the the exponential function poses problems. Therefore, we will switch to spherical coordinates, as the region described is exactly the sphere centered at the origin with radius 1. For the bounds given in terms of x, y, and z, we convert everything to cylindrical coordinates as the following:

$$\begin{aligned} x &= r\sin(\phi)\cos(\theta) \\ y &= r\sin(\phi)\sin(\theta) \\ z &= r\cos(\phi) \\ dV &= r^2\sin(\phi)\ drd\phi d\theta \\ 0 &\leq r \leq 1 \\ 0 &\leq \theta \leq 2\pi \\ 0 &\leq \phi \leq \pi \end{aligned}$$

From here we can set up the integral. We then get

$$\begin{aligned}
\iiint_W e^{\sqrt{(x^2+y^2+z^2)^3}}\ dV &= \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2 e^{r^3}\sin(\phi)\ drd\phi d\theta \\
&= \left(\int_0^1 r^2 e^{r^3}\ dr\right)\left(\int_0^{2\pi}\ d\theta\right)\left(\int_0^{\pi}\sin(\phi)\ d\phi\right) \\
&= \left(\left.\frac{e^{r^3}}{3}\right|_0^1\ dz\right)(2\pi)\left(-\cos(\phi)|_0^{\pi}\right) \\
&= \frac{4}{3}\pi(e-1)
\end{aligned}$$

□

3. Find the solid enclosed by the hyperboloid $-x^2 - y^2 + z^2 = 1$ and the plane $z = 2$.

***Solution*:** Note the the hyperboloid piece is the one in the upper half space, and opens upwards from $z = 1$. First, we write out the integral in Cartesian coordinates. Note that the cross-sections of the region in the xy-plane are circles. We can determine this by some algebra

$$\begin{aligned}
z &= \sqrt{1+x^2+y^2} \\
2 &= \sqrt{1+x^2+y^2} \\
4 &= 1+x^2+y^2 \\
3 &= x^2+y^2
\end{aligned}$$

So the largest circle (that sits on the plane $z = 2$) is given by the above formula, so the radius is $r = \sqrt{3}$. So we write the volume integral as

$$\iiint_W 1\ dV = \int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{\sqrt{1+x^2+y^2}}^{2} 1\ dzdydx$$

This computation will not be very clean if we do this in Cartesian coordinates. Using cylindrical coordinates, we have that

$$\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
z &= z \\
dV &= r\ dzdrd\theta \\
0 \le r &\le \sqrt{3} \\
0 \le \theta &\le 2\pi \\
\sqrt{r^2+1} \le z &\le 2
\end{aligned}$$

So now we have

$$\begin{aligned}
\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_{\sqrt{1+x^2+y^2}}^{2} 1\ dzdydx &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_{\sqrt{r^2+1}}^{2} r\ dzdrd\theta \\
&= 2\pi\int_0^{\sqrt{3}} r(2-\sqrt{r^2+1})dr \\
&= 2\pi\left.\left(r^2 - \frac{1}{3}(r^2+1)^{3/2}\right)\right|_0^{\sqrt{3}} \\
&= \frac{4}{3}\pi
\end{aligned}$$

□

4. Evaluate the following by using cylindrical coordinates.

$$\iiint_W (x^2+y^2+z^2)^{-1/2}\ dxdydz$$

where $W = \left\{ (x,y,z) \;\middle|\; \frac{1}{4} \le z \le 1, x^2+y^2+z^2 \le 1 \right\}$.

***Solution*:** Note that the region W being described is the upper hemisphere of the unit sphere. To use cylindrical coordinates we use

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta) \\ z &= z \end{aligned}$$

Also, the key to this problem is rewriting the r bounds. Since $z = z$, then $\frac{1}{4} \le z \le 1$. Since we are using the whole hemisphere, we take $0 \le \theta \le 2\pi$. Now notice that the sphere is given by $x^2+y^2+z^2 = 1$, or in cylindrical coordinates, $r^2+z^2 = 1$. Solving this for r, we get that $r = \sqrt{1-z^2}$, where we have taken the positive one, as we are concerned with the top hemisphere. Therefore, in cylindrical coordinates, r gives the distance from the z-axis, so it is changing as you increase z from $\frac{1}{4}$ to 1. So, we must have $0 \le r \le \sqrt{1-z^2}$. Note that when writing the bounds for r you need the function to be of some other variable, so in this case $r = f(z)$. Now we put the pieces together

$$\begin{aligned} \iiint_W (x^2+y^2+z^2)^{-1/2}\ dx\ dy\ dz &= \int_0^{2\pi}\int_{1/4}^1\int_0^{\sqrt{1-z^2}} (r^2+z^2)^{-1/2} r\ dr\ dz\ d\theta \\ &= 2\pi \int_{1/4}^1 \sqrt{r^2+z^2}\Big|_{r=0}^{r=\sqrt{1-z^2}}\ dz \\ &= 2\pi \int_{1/4}^1 (1-z)\ dz \\ &= 2\pi\ (z - \frac{z^2}{2})\Big|_{1/4}^1 \\ &= 2\pi\left(\frac{9}{32}\right) \\ &= \frac{9}{16}\pi \end{aligned}$$

□

5. Evaluate

$$\iiint_W \frac{1}{(x^2+y^2+z^2)^{3/2}} dx\ dy\ dz$$

where W is the solid bounded by the two spheres $x^2+y^2+z^2 = a^2$ and $x^2+y^2+z^2 = b^2$, where $0 < b < a$.

***Solution*:** Note that the region W being described is region that is in between the two spheres of different radii. We can use spherical coordinates, and we can describe the region with the inequalities

$$\begin{aligned}
x &= r\sin(\phi)\cos(\theta) \\
y &= r\sin(\phi)\sin(\theta) \\
z &= r\cos(\phi) \\
dV &= r^2\sin(\phi)\ dr\ d\phi\ d\theta \\
b &\le r \le a \\
0 &\le \phi \le \pi \\
0 &\le \theta \le 2\pi
\end{aligned}$$

where the bounds follow since we have the whole sphere, with radii bounded between b and a, as the directions state. Now we put the pieces together

$$\begin{aligned}
\iiint_W \frac{1}{(x^2+y^2+z^2)^{3/2}} dx\ dy\ dz &= \int_0^{2\pi}\int_0^{\pi}\int_b^a \frac{1}{(r^2)^{3/2}} r^2\sin(\phi)\ dr\ d\phi\ d\theta \\
&= \int_0^{2\pi}\int_0^{\pi}\int_b^a \frac{1}{r}\sin(\phi)\ dr\ d\phi\ d\theta \\
&= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin(\phi)\ d\phi\right)\left(\int_b^a \frac{1}{r}\ dr\right) \\
&= 2\pi\left(-\cos(\pi)+\cos(0)\right)\left(\ln(a)-\ln(b)\right) \\
&= 4\pi\ln\left(\frac{a}{b}\right)
\end{aligned}$$

$\square$

6. Using spherical coordinates, compute the integral of $f(\rho,\phi,\theta) = \frac{1}{rho}$ over the region in the first octant of $\mathbb{R}^3$, which is bounded by the cones $\phi = \frac{\pi}{4}$, $\phi = \arctan(2)$, and the sphere $\rho = \sqrt{2}$.

***Solution*:** We can use spherical coordinates, and we can describe the region with the inequalities

$$\begin{aligned}
x &= \rho\sin(\phi)\cos(\theta) \\
y &= \rho\sin(\phi)\sin(\theta) \\
z &= \rho\cos(\phi) \\
dV &= \rho^2\sin(\phi)\ dr\ d\phi\ d\theta \\
0 &\le \rho \le \sqrt{2} \\
\frac{\pi}{4} &\le \phi \le \arctan(2) \\
0 &\le \theta \le \frac{\pi}{2}
\end{aligned}$$

where the bounds follow since we have the whole sphere, with radii bounded between b and a, as the directions state. Now we put the pieces together

$$\begin{aligned}
\iiint_W \frac{1}{\rho}\rho^2 \sin(\phi)\ d\rho\ d\phi\ d\theta &= \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{4}}^{\arctan(2)} \int_0^{\sqrt{2}} \rho \sin(\phi)\ dr\ d\phi\ d\theta \\
&= \left(\int_0^{\frac{\pi}{2}} d\theta\right) \left(\int_{\frac{\pi}{4}}^{\arctan(2)} \sin(\phi)\ d\phi\right) \left(\int_0^{\sqrt{2}} \rho\ d\rho\right) \\
&= \frac{\pi}{2}\left(-\cos\left(\arctan(2)\right) + \cos\left(\frac{\pi}{4}\right)\right)\left(\frac{(\sqrt{2})^2}{2}\right) \\
&= \frac{\pi}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{5}}{5}\right)
\end{aligned}$$

where we have deduced the value of $\cos(\arctan(2))$ by noting that

$$\begin{aligned}
\alpha = \arctan(2) \quad &\Rightarrow \quad \tan(\alpha) = \frac{2}{1} = \frac{\text{opp}}{\text{adj}} \\
&\Rightarrow \quad \text{hyp} = \sqrt{2^2 + 1^2} = \sqrt{5} \\
&\Rightarrow \quad \cos(\alpha) = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}
\end{aligned}$$

□