MATH 010B - Spring 2018

Worked Problems - Section 7.1

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1. Find a parameterization for a curve C which goes along the circle of radius 8 from $(8, 0)$ to $(-8, 0)$ above the x-axis, and then a straight line on the x axis from $(-8, 0)$ to $(8, 0)$.

***Solution*:** We will write the circular part of the curve as $C_1(t)$ and the straight line as $C_2(t)$. So assume that $C_1(t)$ is traversed for the times $0 \leq t \leq 1$, and $C_2(t)$ is traversed for times $1 \leq t \leq 2$. Now, the circle we can represent in terms of polar coordinates via the formulas $x(t) = r\cos(\theta(t))$ and $y(t) = r\sin(\theta(t))$, where $\theta(t)$ and r must be determined. From the picture, we know that the radius is 8, so $r = 8$. Now, if we want to be at $(8, 0)$ at $t = 0$ and at $(-8, 0)$ at $t = 1$, we can take $\theta(t) = \pi t$, therefore we have

$$C_1(t) = (8\cos(\pi t), 8\sin(\pi t)).$$

For the curve $C_2(t)$, notice that the y value on the line is always zero. So how do we find the equation? For $t = 1$ we need $x = -8$ and for $t = 2$ we need $x = 8$. So the slope of the line with these requirements is $x(t) = 16t - 24$. Therefore,

$$C_2(t) = (16t - 24, 0).$$

□

2. Find a parameterization for the given piecewise smooth curve in $\mathbb{R}^3$. The intersection of the plane $z = 7$ with the elliptical cylinder $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

***Solution*:** First we will determine the curve being described in the problem statement. The curve $C(t)$ is an ellipse in the xy-plane, and is on the plane $z = 7$. Therefore, the z component of the curve must always be 7. Note that the ellipse can be written in coordinates similar to a circle, where the radii are different sizes, in this case, 2 in the x variable, and 3 in the y variable. Therefore, we get the parameterization

$$C(t) = (2\cos(2\pi t), 3\sin(2\pi t), 7) \qquad t \in [0, 1],$$

where we choose $2\pi t$ as the argument since we have t ranging from 0 to 1. □

3. Evaluate

$$\int_C f(x, y, z)\ ds$$

for $f(x, y, z) = e^{\sqrt{z}}$, where C is the curve parameterized as $\mathbf{x}(t) = (7, 9, t^2)$, for $0 \leq t \leq 1$.

***Solution*:** Recall that the formula for the path integral of a scalar function is given as

$$\int_C f\ ds = \int_a^b f(\mathbf{x}(t))\ ||\mathbf{x}'(t)||\ dt$$

So we must figure out the other pieces. They are computed as follows:

$$\begin{aligned} f(\mathbf{x}(t)) &= e^{\sqrt{t^2}} = e^t \\ \mathbf{x}'(t) &= (\frac{d}{dt}7, \frac{d}{dt}9, \frac{d}{dt}t^2) = (0,0,2t) \\ ||\mathbf{x}'(t)|| &= \sqrt{0^2+0^2+(2t)^2} = 4t \end{aligned}$$

Now we just compute by putting everything together

$$\begin{aligned} \int_C f\ ds &= \int_a^b f(\mathbf{x}(t))\ ||\mathbf{x}'(t)||\ dt \\ &= \int_0^1 2te^t\ dt \\ &= 2e^t(t-1)\Big|_0^1 \\ &= 2 \end{aligned}$$

□

4. Find the path integral of $f(x,y) = y$ over the graph $y = \sqrt{49-x^2}$ for $-7 \le x \le 7$.

***Solution*:** Here we can use the modified formula for finding the path integral over a graph $y = g(x)$, which is given as

$$\begin{aligned} \int_C f\ ds &= \int_a^b f(x,g(x))\sqrt{1+[g'(x)]^2}\ dx \\ &= \int_{-7}^7 \sqrt{49-x^2}\sqrt{1+\left[\frac{-x}{\sqrt{49-x^2}}\right]^2}\ dx \\ &= \int_{-7}^7 \sqrt{49-x^2}\sqrt{\frac{49}{49-x^2}}\ dx \\ &= 7\int_{-7}^7 1\ dx \\ &= 7(14) = 98 \end{aligned}$$

□

5. Find the path integral of $f(x,y) = 5y^2$ over the graph $y = e^x$ for $0 \le x \le 1$.

***Solution*:** Here we can use the modified formula for finding the path integral over a graph $y = g(x)$ from the previous problem, which is given as

$$\begin{aligned} \int_C f\ ds &= \int_a^b f(x,y(x))\sqrt{1+[y'(x)]^2}\ dx \\ &= \int_0^1 5e^{2x}\sqrt{1+e^{2x}}\ dx \qquad \text{let } u = 1+e^{2x},\ du = 2e^{2x}\ dx \\ &= \frac{5}{2}\int_2^{1+e^2} \sqrt{u}\ du \\ &= \frac{5}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_2^{1+e^2} \\ &= \frac{5}{3}\left(\left(1+e^2\right)^{3/2} - 2\sqrt{2}\right) \end{aligned}$$

□