**MATH 010B - Spring 2018**

Worked Problems - Section 7.2

**1.** Evaluate

$$\int_C \mathbf{F}(x, y, z)\ ds$$

where $\mathbf{F}(x, y, z) = (x, y, z)$, and $C$ is parameterized by $\mathbf{x}(t) = (2\cos(t), 2\sin(t), 0)$, for $0 \le t \le 2\pi$.

***Solution*:** Recall that the formula for the path integral of a vector field $\mathbf{F}$ is given as

$$\int_C \mathbf{F}\ ds = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\ dt$$

So we must figure out the other pieces. They are computed as follows:

$$\begin{aligned}
\mathbf{F}(\mathbf{x}(t)) &= (2\cos(t), 2\sin(t), 0) \\
\mathbf{x}'(t) &= (-2\sin(t), 2\cos(t), 0) \\
\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) &= (2\cos(t), 2\sin(t), 0) \cdot (-2\sin(t), 2\cos(t), 0) \\
&= -4\cos(t)\sin(t) + 4\cos(t)\sin(t) \\
&= 0
\end{aligned}$$

Now we just compute by putting everything together

$$\begin{aligned}
\int_C \mathbf{F}\ ds &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\ dt \\
&= \int_0^{2\pi} 0\ dt \\
&= 0
\end{aligned}$$

□

**2.** Evaluate

$$\int_C 2xyz\ dx + x^2z\ dy + x^2y\ dz$$

where $C$ connects $(1,2,4)$ to $(1,4,16)$.

***Solution*:** Use a parameterization for a straight line between the points for $0 \leq t \leq 1$. To do this, follow a similar procedure as the straight line in Section 7.1, number 1. Thus we have

$$\begin{aligned} \mathbf{x}(t) &= (1, 2t+2, 12t+4) \\ \mathbf{x}'(t) &= (0, 2, 12) \end{aligned}$$

Note that given $\mathbf{x}(t) = (x,y,z)$, then the derivative is given as $\mathbf{x}'(t) = (dx, dy, dz)$. Now just substitute out

$$\begin{aligned} \int_C 2xyz\ dx + x^2z\ dy + x^2y\ dz &= \int_0^1 (0 + (1)(12t+4)(2) + (1)(2t+2)(12))\ dt \\ &= \int_0^1 48t + 32\ dt \\ &= 24t^2 + 32\Big|_0^1 \\ &= 56 \end{aligned}$$

□

**3.** Evaluate

$$\int_C x^2\ dx + xy\ dy + \ dz$$

where $C$ is the parabola $z = x^2$ from $(-2,0,4)$ to $(2,0,4)$.

***Solution*:** Note that the parabola $z = x^2$ is a parabola in the $xz$-plane, where the $y$ component is always zero. If we let $x = t$ for $t \in [-2,2]$, then $z = t^2$. Therefore, we get

$$\begin{aligned} \mathbf{x}(t) &= (t, 0, t^2) \\ \mathbf{x}'(t) &= (1, 0, 2t) \end{aligned}$$

Note that given $\mathbf{x}(t) = (x,y,z)$, then the derivative is given as $\mathbf{x}'(t) = (dx, dy, dz)$. Now just substitute out

$$\begin{aligned} \int_C x^2\ dx + xy\ dy + \ dz &= \int_{-2}^2 ((t^2)(1) + (1)(0)(0) + 2t)\ dt \\ &= \int_{-2}^2 (t^2 + 2t)\ dt \\ &= \frac{1}{3}t^3 + t^2\Big|_{-2}^2 \\ &= \frac{16}{3} \end{aligned}$$

□

**4.** Consider the force field $\mathbf{F}(x, y, z) = (x, y, z)$. Compute the work done in moving a particle along the parabola $y = x^2$, $z = 0$, from $x = -1$ to $x = 3$.

***Solution*:** Note that the parabola $y = x^2$ is a parabola in the $xy$-plane, where the $z$ component is always zero. If we let $x = t$ for $t \in [-1, 3]$, then $y = t^2$. Therefore, we get

$$\begin{aligned} \mathbf{x}(t) &= (t, t^2, 0) \\ \mathbf{x}'(t) &= (1, 2t, 0) \\ \mathbf{F}(\mathbf{x}(t)) &= (t, t^2, 0) \end{aligned}$$

Now we just use the formula

$$\begin{aligned} W = \int_C \mathbf{F}\ ds &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\ dt \\ &= \int_{-1}^3 (t, t^2, 0) \cdot (1, 2t, 0)\ dt \\ &= \int_{-1}^3 (t + 2t^3)\ dt \\ &= \frac{1}{2}t^2 + \frac{1}{2}t^4 \Big|_{-1}^3 \\ &= \frac{9}{2} + \frac{81}{2} - \frac{1}{2} - \frac{1}{2} = \frac{88}{2} \\ &= 44 \end{aligned}$$

□

**5.** Given

$$\nabla f(x, y, z) = (2xyze^{x^2}, ze^{x^2}, ye^{x^2})$$

and $f(0, 0, 0) = 1$, determine the value of $f(3, 3, 2)$.

***Solution*:** Recall that the gradient of $f$ is defined as

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (2xyze^{x^2}, ze^{x^2}, ye^{x^2})$$

so we then know that the following equation holds

$$\frac{\partial f}{\partial x} = 2xyze^{x^2}.$$

Now we can integrate both sides of the above equation to figure out what the general $f$ is, recalling that the integration "constant" can be a function $y$ and $z$. Therefore, we have that

$$f(x, y, z) = yze^{x^2} + g(y, z)$$

Now taking partial derivatives of the general $f$ with respect to $y$ and $z$, we get

$$\begin{aligned} \frac{\partial f}{\partial y} &= ze^{x^2} + g_y(y, z) \\ \frac{\partial f}{\partial x} &= ye^{x^2} + g_z(y, z). \end{aligned}$$

If you match the terms above with the original gradient that was given in the problem, we must have that

$$g_y(y, z) = 0$$
$$g_z(y, z) = 0$$

and by integrating, we get that

$$g(y, z) = h(z)$$
$$g(y, z) = \tilde{h}(y)$$

So the above condition states that $g$ must be *strictly* a function of $z$ **and** *strictly* a function of $y$. The only way this is possible is if $g(y, z) = C$, where $C$ is a constant. So we have that

$$f(x, y, z) = yze^{x^2} + C$$

Now we use the initial condition that is given to show that

$$f(0, 0, 0) = 1 = 0 + C$$

so that we find $C = 1$. Therefore, our function is $f(x, y, z) = yze^{x^2} + 1$, and $f(x, y, z) = 6e^9 + 1$. □