MATH 010B - Spring 2018

Worked Problems - Section 7.3

1. Find an equation for the plane tangent to the given surface at the specified point.

$$x = 3u, \qquad y = u^2 + v, \qquad z = v^2, \qquad \text{at } (0, 3, 9)$$

***Solution*:** First we compute the tangent vectors of the parameterized surface:

$$\mathbf{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (3, 2u, 0)$$
$$\mathbf{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (0, 1, 2v)$$

Then we compute the cross-product of the tangent vectors

$$\mathbf{T}_u \times \mathbf{T}_v = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 3 & 2u & 0 \\ 0 & 1 & 2v \end{vmatrix} = \widehat{i}(4uv) - \widehat{j}(6v) + \widehat{k}(3)$$

which means that we have $\mathbf{T}_u \times \mathbf{T}_v = (4uv, -6v, 3)$. Therefore, at the point $(0, 3, 9)$, the parametric equations are

$$0 = 3u$$
$$3 = u^2 + v$$
$$9 = v^2$$

And solving this system of equations gives us that $u = 0$ and $v = 3$. Thus we compute the normal vector $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v = (4uv, -6v, 3) = (0, -18, 3) = (N_x, N_y, N_z)$, and then write the equation of the plane

$$N_x(x - x_0) + N_y(y - y_0) + N_z(z - z_0) = 0$$
$$0(x - 0) - 18(y - 3) + 3(z - 9) = 0$$
$$-18(y - 3) + 3(z - 9) = 0$$

□

2. Consider the following surface parametrization.

$$x = 4\cos(\theta)\sin(\phi), \qquad y = 2\sin(\theta)\sin(\phi), \qquad z = \cos(\phi)$$

Find an expression for a unit vector, **N**, normal to the surface at the image of a point (u, v) for $\theta \in [0, 2\pi]$ and $\phi \in [0, \phi]$. Identify the surface.

Solution: First notice that this parameterization looks a lot like the spherical coordinates formula, except that the radii in the x and y components are different. This would signal to us that this has some elliptical shape. Essentially we get a unit sphere that is stretched 4 units along the x-axis, and 2 units along the y-axis, which would give us an ellipsoid. First we compute the tangent vectors of the parameterized surface:

$$\begin{aligned}
\mathbf{T}_\theta &= \left(\frac{\partial x}{\partial \theta}, \frac{\partial y}{\partial \theta}, \frac{\partial z}{\partial \theta}\right) = (-4\sin(\theta)\sin(\phi), 2\cos(\theta)\sin(\phi), 0) \\
\mathbf{T}_\phi &= \left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi}, \frac{\partial z}{\partial \phi}\right) = (4\cos(\theta)\cos(\phi), 2\sin(\theta)\cos(\phi), -\sin(\phi))
\end{aligned}$$

Then we compute the cross-product of the tangent vectors

$$\begin{aligned}
\mathbf{T}_\theta \times \mathbf{T}_\phi &= \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ -4\sin(\theta)\sin(\phi) & 2\cos(\theta)\sin(\phi) & 0 \\ 4\cos(\theta)\cos(\phi) & 2\sin(\theta)\cos(\phi) & -\sin(\phi) \end{vmatrix} \\
&= \widehat{i}(-2\cos(\theta)\sin^2(\phi)) - \widehat{j}(4\sin(\theta)\sin^2(\phi)) \\
&+ \widehat{k}(-8\sin^2(\theta)\sin(\phi)\cos(\phi) - 8\cos^2(\theta)\sin(\phi)\cos(\phi)) \\
&= \widehat{i}(-2\cos(\theta)\sin^2(\phi)) + \widehat{j}(-4\sin(\theta)\sin^2(\phi)) + \widehat{k}(-8\sin(\phi)\cos(\phi)) \\
&= \sin(\phi)\left[\widehat{i}(-2\cos(\theta)\sin(\phi)) + \widehat{j}(-4\sin(\theta)\sin(\phi)) + \widehat{k}(-8\cos(\phi))\right]
\end{aligned}$$

Then we have to compute the length of the vector, then divide by the length to get the unit normal. So,

$$\begin{aligned}
||\mathbf{T}_\theta \times \mathbf{T}_\phi|| &= \sqrt{N_x^2 + N_y^2 + N_z^2} \\
&= \sqrt{\sin^2(\phi)\left(4\cos^2(\theta)\sin^2(\phi)\right) + 16\sin^2(\theta)\sin^2(\phi) + 64\cos^2(\phi)\big)} \\
&= |\sin(\phi)|\sqrt{4\sin^2(\phi) + 12\sin^2(\theta)\sin^2(\phi) + 64\cos^2(\phi)} \\
&= \sin(\phi)\sqrt{4 + 12\sin^2(\theta)\sin^2(\phi) + 60\cos^2(\phi)}
\end{aligned}$$

where we drop absolute value on $\sin(\phi)$ since $0 \leq \phi \leq \pi$, and use the trig identity $\cos^2(x) + \sin^2(x) = 1$ two times. Then we compute the unit normal

$$\begin{aligned}
\mathbf{N} &= \frac{\mathbf{T}_\theta \times \mathbf{T}_\phi}{||\mathbf{T}_\theta \times \mathbf{T}_\phi||} \\
&= \frac{\sin(\phi)\left[\widehat{i}(-2\cos(\theta)\sin(\phi)) + \widehat{j}(-4\sin(\theta)\sin(\phi)) + \widehat{k}(-8\cos(\phi))\right]}{\sin(\phi)\sqrt{4 + 12\sin^2(\theta)\sin^2(\phi) + 60\cos^2(\phi)}} \\
&= \frac{(-2\cos(\theta)\sin(\phi), -4\sin(\theta)\sin(\phi), -8\cos(\phi))}{\sqrt{4 + 12\sin^2(\theta)\sin^2(\phi) + 60\cos^2(\phi)}}
\end{aligned}$$

□