MATH 010B - Spring 2018

Worked Problems - Section 7.5

1. Evaluate

$$\iint_S x + y \; ds$$

where S is the surface parameterized by $\mathbf{T}(u,v) = (2u\cos(v), 2u\sin(v), u)$ for $u \in [0,4]$ and $v \in [0,\pi]$.

***Solution*:** Recall that the formula for scalar functions over surfaces is given by

$$\iint_S f(x,y,z)\; ds = \iint_D f(\mathbf{T}(u,v))||\mathbf{T}_u \times \mathbf{T}_v||\; du\; dv$$

Now we must compute all of the pieces. They are given as

$$\begin{aligned}
\mathbf{T}_u &= (2\cos(v), 2\sin(v), 1)\\
\mathbf{T}_v &= (-2u\sin(v), 2u\cos(v), 0)\\
||\mathbf{T}_u \times \mathbf{T}_v|| &= \sqrt{4u^2\cos^2(v) + 4u^2\sin^2(v) + 16u^2} = \sqrt{20u^2} = 2\sqrt{5}u\\
f(\mathbf{T}(u,v)) &= 2u(\sin(v) + \cos(v))
\end{aligned}$$

Now insert all of the pieces into the formula to get

$$\begin{aligned}
\iint_S x + y\; ds &= \int_0^\pi \int_0^4 (2u(\sin(v) + \cos(v)))\; 2\sqrt{5}u\; du\; dv\\
&= 4\sqrt{5} \int_0^\pi \int_0^4 (u^2(\sin(v) + \cos(v)))\; dudv\\
&= 4\sqrt{5} \left(\int_0^\pi (\sin(v) + \cos(v)))\; dv\right)\left(\int_0^4 u^2\; du\right)\\
&= 4\sqrt{5}\,(-\cos(v) + \sin(v)|_0^\pi) \left(\frac{u^3}{3}\bigg|_0^4\right)\\
&= 4\sqrt{5}(2)\left(\frac{64}{3}\right) = \frac{512\sqrt{5}}{3}
\end{aligned}$$

□

2. Find the average value of $f(x,y,z) = x + z^2$ on the surface S for the truncated cone $z^2 = x^2 + y^2$, with $1 \leq z \leq 9$.

***Solution*:** To find the average value, we need to compute two integrals, one where $f(x,y,z) = 1$ and the other where $f(x,y,z) = x + z^2$. To do both, we need to compute the components in the formula for scalar functions given in the previous question. These pieces are the parameterization $\mathbf{T}(u,v)$, and its derivatives, and its norm.

$$\begin{aligned}
\mathbf{T}(u,v) &= (u\cos(v), u\sin(v), u)\\
\mathbf{T}_u &= (\cos(v), \sin(v), 1)\\
\mathbf{T}_v &= (-u\sin(v), u\cos(v), 0)\\
\mathbf{T}_u \times \mathbf{T}_v &= (-u\cos(v), -u\sin(v), u(\cos^2(v) + \sin^2(v)))\\
||\mathbf{T}_u \times \mathbf{T}_v|| &= \sqrt{u^2\cos^2(v) + u^2\sin^2(v) + u^2} = \sqrt{2u^2} = \sqrt{2}u
\end{aligned}$$

where $u \in [1,9]$ and $[0, 2\pi]$. Now, first we compute where $f(x,y,z) = 1$. So insert all of the pieces into the formula to get

$$\begin{aligned}
\iint_S 1\ ds &= \int_0^{2\pi} \int_1^9 \sqrt{2}u\ du\ dv \\
&= 2\pi \left(\sqrt{2}\frac{u^2}{2}\Big|_1^9 \right) \\
&= 80\pi\sqrt{2}
\end{aligned}$$

Now we do the computation for $f(x,y,z) = x + z^2$. We then have

$$\begin{aligned}
\iint_S x + z^2\ ds &= \int_0^{2\pi} \int_1^9 (u\cos(v) + u^2)\sqrt{2}u\ du\ dv \\
&= \sqrt{2}\int_0^{2\pi} \int_1^9 (u^2\cos(v) + u^3)\ du\ dv \\
&= 3280\pi\sqrt{2}
\end{aligned}$$

To find the average value, we just take the ratio of the two solutions we got above

$$\text{Average Value} = \frac{3280\pi\sqrt{2}}{80\pi\sqrt{2}} = 41$$

$\square$

3. Evaluate

$$\iint_S x + z\ ds$$

where S is the is the part of the cylinder $y^2 + z^2 = 4$ with $x \in [0, 2]$.

***Solution*:** Again, we follow the same procedure in number 1.

$$\begin{aligned}
\mathbf{T}(u,v) &= (u, 2\cos(v), 2\sin(v)) \\
\mathbf{T}_u &= (1,0,0) \\
\mathbf{T}_v &= (0, -2\sin(v), 2\cos(v)) \\
\mathbf{T}_u \times \mathbf{T}_v &= (0, -2\cos(v), -2\sin(v)) \\
||\mathbf{T}_u \times \mathbf{T}_v|| &= \sqrt{4\cos^2(v) + 4\sin^2(v)} = \sqrt{4} = 2 \\
f(\mathbf{T}(u,v)) &= u + 2\sin(v)
\end{aligned}$$

where $u \in [0,2]$ and $v \in [0, 2\pi]$. Now insert all of the pieces into the formula to get

$$\begin{aligned}
\iint_S x + z\ ds &= \int_0^{2\pi} \int_0^2 (u + 2\sin(v))\ 2\ du\ dv \\
&= 2\int_0^{2\pi} \int_0^2 \left(\frac{u^2}{2} + 2u\sin(v) \right)\Big|_0^2\ dv \\
&= 2\int_0^{2\pi} (2 + 4\sin(v))\ dv \\
&= 4\ (v - 2\cos(v))|_0^{2\pi} \\
&= 4(2\pi - 2 + 2) = 8\pi
\end{aligned}$$

$\square$

4. Evaluate

$$\iint_S 9xy\ ds$$

where S is the surface of the tetrahedron with sides $y = 0, z = 0, x + z = 1$, and $x = y$.

***Solution*:** For this problem, consider the four sides of the tetrahedron as: S_1 being $y = 0$, C_2 being $z = 0$, S_3 being $x + z = 1$, and S_4 being $x = y$. We can then compute the pieces individually, as

$$\iint_S 9xy\ ds = \iint_{S_1} 9xy\ ds + \iint_{S_2} 9xy\ ds + \iint_{S_3} 9xy\ ds + \iint_{S_4} 9xy\ ds$$

For the first face, S_1, note that $y \equiv 0$ on this face, so then we simply get

$$\iint_{S_1} 9xy\ ds = 0$$

On the second face, S_2, note that on the plane $z = 0$, is a triangular region that is given in the xy-plane by the equations $x = 1$, $y = 0$, and $x = y$. Therefore, we have the integration over S_2 is

$$\begin{aligned}
\iint_{S_2} 9xy\ ds &= \int_0^1 \int_y^1 9xy\ dx\ dy \\
&= 9\int_0^1 \left(\frac{1}{2}x^2y\right)\Big|_{x=y}^{x=1}\ dy \\
&= \frac{9}{2}\int_0^1 \left(y - y^3\right)\ dy \\
&= \frac{9}{2}\left(\frac{1}{2}y^2 - \frac{1}{4}y^4\right)\Big|_0^1 \\
&= \frac{9}{8}
\end{aligned}$$

On the third face, S_3 note that $x + z = 1$ implies that $z = 1 - x$. So we take the following parameterization

$$\begin{aligned}
\mathbf{T}(u, v) &= (u, v, 1 - u) \\
\mathbf{T}_u &= (1, 0, -1) \\
\mathbf{T}_v &= (0, 1, 0) \\
\mathbf{T}_u \times \mathbf{T}_v &= (1, 0, 1) \\
||\mathbf{T}_u \times \mathbf{T}_v|| &= \sqrt{2} \\
f(\mathbf{T}(u, v)) &= 9uv
\end{aligned}$$

for $u \in [v, 1]$, and $v \in [0, 1]$. Now insert all of the pieces into the formula to get

$$\begin{aligned}
\iint_{S_3} 9xy\ ds &= \int_0^1 \int_v^1 9\sqrt{2}uv\ du\ dv \\
&= 9\sqrt{2}\int_0^1 \left(\frac{1}{2}u^2v\right)\Big|_{u=v}^{u=1}\ dv \\
&= \frac{9\sqrt{2}}{2}\int_0^1 \left(v - v^3\right)\ dv \\
&= \frac{9\sqrt{2}}{2}\left(\frac{1}{2}v^2 - \frac{1}{4}v^4\right)\Big|_0^1 \\
&= \frac{9\sqrt{2}}{8}
\end{aligned}$$

On the fourth face, S_4 note that $x = y$, and $x + z = 1$ implies that $x = 1 - z$. So we take the following parameterization

$$\begin{aligned}
\mathbf{T}(u, v) &= (u, u, v) \\
\mathbf{T}_u &= (1, 1, 0) \\
\mathbf{T}_v &= (0, 0, 1) \\
\mathbf{T}_u \times \mathbf{T}_v &= (1, -1, 0) \\
||\mathbf{T}_u \times \mathbf{T}_v|| &= \sqrt{2} \\
f(\mathbf{T}(u, v)) &= 9u^2
\end{aligned}$$

for $u \in [0, x] = [0, 1 - z] = [0, 1 - v]$, and $v \in [0, 1]$. Now insert all of the pieces into the formula to get

$$\begin{aligned}
\iint_{S_3} 9xy\ ds &= \int_0^1 \int_0^{1-v} 9\sqrt{2}u^2\ du\ dv \\
&= 9\sqrt{2} \int_0^1 \left(\frac{1}{3}u^3 \right) \Bigg|_{u=0}^{u=1-v}\ dv \\
&= \frac{9\sqrt{2}}{3} \int_0^1 (1 - v)^3\ dv \\
&= 3\sqrt{2} \left(-\frac{1}{4}(1 - v)^4 \right) \Bigg|_0^1 \\
&= \frac{3\sqrt{2}}{4}
\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}
\iint_S 9xy\ ds &= \iint_{S_1} 9xy\ ds + \iint_{S_2} 9xy\ ds + \iint_{S_3} 9xy\ ds + \iint_{S_4} 9xy\ ds \\
&= 0 + \frac{9}{8} + \frac{9\sqrt{2}}{8} + \frac{3\sqrt{2}}{4} \\
&= \frac{15\sqrt{2} + 9}{8}
\end{aligned}$$

$\square$

5. (a) Compute the area of the portion of the cone $x^2 + y^2 = z^2$ with $z \geq 0$ that is inside the sphere $x^2 + y^2 + z^2 = 2Rz$, where R is a positive constant.
(b) What is the area of that portion of the sphere that is inside the cone?

***Solution*:** (a) For this problem, we use the formula for surface area over a graph given by

$$\iint_S f(x, y, z)\ ds = \iint_D f(x, y, g(x, y)) \sqrt{1 + \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2}\ dx\ dy$$

For the area of the region, we take $f(x, y, z) = 1$. Note that $x^2 + y^2 = z^2$ is equal to $z = \sqrt{x^2 + y^2}$, since we only consider $z \geq 0$. Now, we take $g(x, y) = \sqrt{x^2 + y^2}$. Also note that the region D is the intersection of the surfaces in the problem, which implies

$$\begin{aligned}
x^2 + y^2 + z^2 &= 2Rz \\
z^2 + z^2 &= 2Rz \\
z &= R
\end{aligned}$$

So the D is the circle with radius R. Therefore, we get the integral

$$
\begin{aligned}
\iint_S f(x,y,z)\ ds &= \iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial g}{\partial x}\right)^2+\left(\frac{\partial g}{\partial y}\right)^2}\ dx\ dy \\
&= \iint_D \sqrt{1+\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2}\ dx\ dy \\
&= \iint_D \sqrt{1+\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}}\ dx\ dy \\
&= \sqrt{2}\iint_D\ dx\ dy \\
&= \sqrt{2}(\text{Area of circle with radius } R) \\
&= \sqrt{2}\pi R^2
\end{aligned}
$$

(b) For this part, see the diagram in the class notes. The part of the sphere that is inside the cone is exactly the upper hemisphere. From geometry, the surface area of a sphere is $4\pi R^2$, but since we only consider the upper hemisphere, we get exactly half this region, which gives $A = 2\pi R^2$. Alternatively, you could set up the surface integral in cylindrical or spherical coordinates to compute the surface area of the upper hemisphere, and you will get the same solution. $\square$