**MATH 010B - Spring 2018**

Worked Problems - Section 7.6

**1.** Consider the closed surface $S$ consisting of the graph $z = 1 - x^2 - y^2$ with $z \geq 0$, and also the unit disc in the $xy$ plane. Give this surface an outer normal. Compute

$$\iint_S \mathbf{F} \cdot d\mathbf{S}$$

where $\mathbf{F}(x, y, z) = (7x, 7y, z)$.

***Solution*:** Recall that the formula for vector fields over surfaces is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{T}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\ du\ dv$$

Now we must compute all of the pieces. They are given as

$$\begin{aligned}
\mathbf{T}(u, v) &= (u, v, 1 - u^2 - v^2) \\
\mathbf{T}_u &= (1, 0, -2u) \\
\mathbf{T}_v &= (0, 1, -2v) \\
\mathbf{T}_u \times \mathbf{T}_v &= (2u, 2v, 1) \\
\mathbf{F}(\mathbf{T}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) &= (7u, 7v, 1 - u^2 - v^2) \cdot (2u, 2v, 1) = 13u^2 + 13v^2 + 1
\end{aligned}$$

So then the problem becomes

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D 13u^2 + 13v^2 + 1\ du\ dv$$

but the bounds here are complicated as we will be integrating over the region $D$ corresponding to the circle of radius 1. This is much easier to do in polar coordinates, so we convert to polar coordinates, then integrate

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D 13u^2 + 13v^2 + 1\ du\ dv \\
&= \int_0^{2\pi} \int_0^1 (13r^2 + 1)r\ dr d\theta \\
&= 2\pi \left( \frac{13}{4} r^4 + \frac{r^2}{2} \right) \Bigg|_0^1 \\
&= 2\pi \frac{15}{4} = \frac{15}{2}\pi
\end{aligned}$$

$\square$

**2.** Let $S$ be the closed surface that consists of the hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$, and its base $x^2 + y^2 \leq 1$, $z = 0$. Let $\mathbf{E}$ be the electric field defined by $\mathbf{E}(x, y, z) = (3x, 3y, 3z)$. Find the electric flux across $S$.

***Solution***: Again, we approach this problem the same as the previous problem. We must compute all of the pieces. They are given as

$$\begin{aligned}
\mathbf{T}(u,v) &= (\cos(u)\sin(v), \sin(u)\sin(v), \cos(v)) \\
\mathbf{T}_u &= (-\sin(u)\sin(v), \cos(u)\sin(v), 0) \\
\mathbf{T}_v &= (\cos(u)\cos(v), \sin(u)\cos(v), -\sin(v)) \\
\mathbf{T}_u \times \mathbf{T}_v &= (-\sin^2(v)\cos(u), -\sin^2(v)\sin(u), -\sin(v)\cos(v)) \\
\mathbf{F}(\mathbf{T}(u,v)) &= (3\cos(u)\sin(v), 3\sin(u)\sin(v), 3\cos(v))
\end{aligned}$$

where $u \in [0, 2\pi]$ and $v \in [0, \pi/2]$. Note that the cross product gives a unit normal that points inward, not outward. To fix this, we can multiply by a minus sign, so that the unit normal points outward. Now we compute the integrand

$$\begin{aligned}
&\mathbf{F}(\mathbf{T}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \\
&= (3\cos(u)\sin(v), 3\sin(u)\sin(v), 3\cos(v)) \cdot (\sin^2(v)\cos(u), \sin^2(v)\sin(u), \sin(v)\cos(v)) \\
&= 3\sin^3(v)\cos^2(u) + 3\sin^3(v)\sin^2(u) + 3\sin(v)\cos^2(v) \\
&= 3\sin^3(v) + 3\sin(v)\cos^2(v)
\end{aligned}$$

Now we can finally compute the integral.

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D 3\sin^3(v) + 3\sin(v)\cos^2(v) \; dudv \\
&= 3\int_0^{\pi/2}\int_0^{2\pi} \sin^3(v) + \sin(v)\cos^2(v) \; dudv \\
&= 6\pi
\end{aligned}$$

□

**3.** Let $S$ be the cylinder $x^2 + y^2 = 9$ with $z \in [0, 1]$. Consider only the lateral surface area of the cylinder. Let $\mathbf{F}(x, y, z) = (5x, -5y, z^2)$. Compute

$$\iint_S \mathbf{F} \cdot \; d\mathbf{S}$$

***Solution***: Again, we approach this problem the same as the previous problem. We must compute all of the pieces. Note that we have a cylinder, so we use a parameterization of cylindrical coordinates. They are given as

$$\begin{aligned}
\mathbf{T}(u,v) &= (3\cos(u), 3\sin(u), v) \\
\mathbf{T}_u &= (-3\sin(u), 3\cos(u), 0) \\
\mathbf{T}_v &= (0, 0, 1) \\
\mathbf{T}_u \times \mathbf{T}_v &= (3\cos(u), 3\sin(u), 0) \\
\mathbf{F}(\mathbf{T}(u,v)) &= (15\cos(u), -15\sin(u), v^2)
\end{aligned}$$

where $u \in [0, 2\pi]$ and $v \in [0, 1]$. You can also test the normal vector to see that it points in the outward direction. Therefore, $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$. Now we compute the integral using the formula

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F}(\mathbf{T}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \; dudv \\
&= \int_0^1 \int_0^{2\pi} (15\cos(u), -15\sin(u), v^2) \cdot (3\cos(u), 3\sin(u), 0) \; du \; dv \\
&= 45 \int_0^1 \int_0^{2\pi} \cos^2(u) - \sin^2(u) \; du \; dv \\
&= 45 \left( \int_0^1 \; dv \right) \left( \int_0^{2\pi} \cos(2u) \; du \right) \\
&= 45 \left( \frac{1}{2} \sin(2u) \right) \Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

□

**4.** Compute

$$\iint_S \mathbf{F} \cdot \; \mathbf{N} \, dA$$

where $\mathbf{F}(x,y,z) = (1, 1, z(x^2+y^2)^2)$ and $S$ is the surface of the cylinder $x^2+y^2 \leq 1$, $0 \leq z \leq 5$.

***Solution*:** For this problem, we will consider three surfaces, the top cap of the cylinder, the bottom cap of the cylinder, and the lateral surface of the cylinder, which we denote $S_1, S_2$, and $S_3$.

For $S_1$, we have the unit disk at $z = 5$, which gives the following pieces

$$\begin{aligned}
\mathbf{T}(u,v) &= (u\cos(v), u\sin(v), 5) \\
\mathbf{T}_u &= (\cos(v), \sin(v), 0) \\
\mathbf{T}_v &= (-u\sin(v), u\cos(v), 0) \\
\mathbf{T}_u \times \mathbf{T}_v &= (0, 0, u) \\
\mathbf{F}(\mathbf{T}(u,v)) &= (1, 1, u^2)
\end{aligned}$$

where $u \in [0,1]$ and $v \in [0, 2\pi]$. You can also test the normal vector to see that it points up. Therefore, $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$. Now we compute the integral using the formula

$$\begin{aligned}
\iint_{S_1} \mathbf{F} \cdot \; \mathbf{N} \, dA &= \iint_{D_1} \mathbf{F}(\mathbf{T}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \; du \; dv \\
&= \int_0^{2\pi} \int_0^1 (1, 1, 5(u^2)^2) \cdot (0, 0, u) \; du \; dv \\
&= 5 \int_0^{2\pi} \int_0^1 u^5 \; du \; dv \\
&= 5(2\pi) \left( \frac{1}{6} \right) \\
&= \frac{5\pi}{3}
\end{aligned}$$

For $S_2$, we have the unit disk at $z = 0$, which gives the following pieces

$$\begin{aligned}
\mathbf{T}(u,v) &= (u\cos(v), u\sin(v), 0) \\
\mathbf{T}_u &= (\cos(v), \sin(v), 0) \\
\mathbf{T}_v &= (-u\sin(v), u\cos(v), 0) \\
\mathbf{T}_u \times \mathbf{T}_v &= (0, 0, u) \\
\mathbf{F}(\mathbf{T}(u,v)) &= (1, 1, u^2)
\end{aligned}$$

where $u \in [0,1]$ and $v \in [0, 2\pi]$. You can also test the normal vector to see that it points up, but we need it to point down. Therefore, $\mathbf{N} = -(\mathbf{T}_u \times \mathbf{T}_v) = (0, 0, -u)$. Now we compute the integral using the formula

$$\begin{aligned}
\iint_{S_2} \mathbf{F} \cdot \mathbf{N}\, dA &= \iint_{D_2} \mathbf{F}(\mathbf{T}(u,v)) \cdot \mathbf{N}\, du\, dv \\
&= \int_0^{2\pi} \int_0^1 (1,1,0) \cdot (0,0,-u)\, du\, dv \\
&= \int_0^{2\pi} \int_0^1 0\, du\, dv \\
&= 0
\end{aligned}$$

For $S_3$, we use the cylinder $x^2 + y^2 \leq 1$, which gives the following pieces

$$\begin{aligned}
\mathbf{T}(u,v) &= (\cos(v), \sin(v), u) \\
\mathbf{T}_u &= (0, 0, 1) \\
\mathbf{T}_v &= (-\sin(v), \cos(v), 0) \\
\mathbf{T}_u \times \mathbf{T}_v &= (\cos(v), \sin(v), 0) \\
\mathbf{F}(\mathbf{T}(u,v)) &= (1, 1, u)
\end{aligned}$$

where $u \in [0,1]$ and $v \in [0, 2\pi]$. You can also test the normal vector to see that it points outwards. Therefore, $\mathbf{N} = \mathbf{T}_u \times \mathbf{T}_v$. Now we compute the integral using the formula

$$\begin{aligned}
\iint_{S_3} \mathbf{F} \cdot \mathbf{N}\, dA &= \iint_{D_3} \mathbf{F}(\mathbf{T}(u,v)) \cdot \mathbf{N}\, du\, dv \\
&= \int_0^{2\pi} \int_0^1 (1,1,u) \cdot (\cos(v), \sin(v), 0)\, du\, dv \\
&= \int_0^{2\pi} \int_0^1 (\cos(v) + \sin(v))\ du\, dv \\
&= 1 \cdot 0 = 0
\end{aligned}$$

Therefore, we combine all the pieces

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{N}\, dA &= \iint_{S_1} \mathbf{F} \cdot \mathbf{N}\, dA + \iint_{S_2} \mathbf{F} \cdot \mathbf{N}\, dA + \iint_{S_3} \mathbf{F} \cdot \mathbf{N}\, dA \\
&= \frac{5\pi}{3} + 0 + 0 \\
&= \frac{5\pi}{3}
\end{aligned}$$

$\square$