MATH 010B - Spring 2018

Worked Problems - Section 8.1

For these questions, recall what Green's Theorem states:

$$\oint_C M\,dx + N\,dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\,dx\,dy$$

which allows us to change the path integral along a curve C to a double integral over a region D. This can drastically simplify the computations of some path integrals to very easy calculations using double integrals.

1. Let D be the triangle in the xy-plane with vertices at $(-2,3),(2,0)$, and $(5,5)$. Describe the boundary ∂D as a piecewise smooth curve, oriented counterclockwise. (Use t as a parameter. Begin the curve at point $(-2,3)$.)

Solution: Just construct linear pieces for each component using a table.

For the line from $(-2,3)$ to $(2,0)$, and for $t \in [0,1]$, use the table

t	x(t)	y(t)
0	-2	3
1	2	0

which gives us the first component as $\mathbf{f}_1(t) = (x(t),y(t)) = (4t-2,-3t+3)$. We do the same for $(2,0)$ to $(5,5)$, and for $t \in [1,2]$

t	x(t)	y(t)
1	2	0
2	5	5

which gives us the second component as $\mathbf{f}_2(t) = (x(t),y(t)) = (3t-1,5t-5)$. We do the same for $(5,5)$ to $(-2,3)$, and for $t \in [2,3]$

t	x(t)	y(t)
2	5	5
3	-2	3

which gives us the third component as $\mathbf{f}_3(t) = (x(t),y(t)) = (-7t+19,-2t+9)$. Therefore, the boundary ∂D can be given as the piecewise curve defined as

$$\mathbf{f}(t) = \begin{cases} \mathbf{f}_1(t) = (4t-2,-3t+3) & \text{for } t \in [0,1] \\ \mathbf{f}_2(t) = (3t-1,5t-5) & \text{for } t \in [1,2] \\ \mathbf{f}_3(t) = (-7t+19,-2t+9) & \text{for } t \in [2,3] \end{cases}$$

□

2. Verify Green's theorem for the indicated region D and boundary ∂D, and functions M and N.

$$D = [-1, 6] \times [-1, 6], \quad M(x, y) = x, \quad N(x, y) = y$$

Solution: Note that the region D is the unit square. The problem tells us to verify Green's theorem, so we must compute both sides. In the question, it asks us to compute over C^+, which I take to mean the positive portion of square D, which is in the upper half plane. Therefore, it has three pieces, which go from $(1, 0)$ to $(1, 1)$, from $(1, 1)$ to $(-1, 1)$, and from $(-1, 1)$ to $(-1, 0)$. The three parameterizations can be given as

$$\begin{aligned} \mathbf{x}_1(t) &= (1, t) \quad \text{for } t \in [0, 1] \\ \mathbf{x}_2(t) &= (-2t + 3, 1) \quad \text{for } t \in [1, 2] \\ \mathbf{x}_3(t) &= (-1, -t + 3) \quad \text{for } t \in [2, 3] \end{aligned}$$

and the derivatives as

$$\begin{aligned} \mathbf{x}_1'(t) &= (0, 1) \quad \text{for } t \in [0, 1] \\ \mathbf{x}_2'(t) &= (-2, 0) \quad \text{for } t \in [1, 2] \\ \mathbf{x}_3'(t) &= (0, -1) \quad \text{for } t \in [2, 3] \end{aligned}$$

Remember that $\mathbf{x}_j(t) = (\text{"}x\text{"}, \text{"}y\text{"})$ and $\mathbf{x}_j'(t) = (\text{"}dx\text{"}, \text{"}dy\text{"})$. Then computing the line integral, we get

$$\begin{aligned} \int_C M(x, y)\ dx + N(x, y)dy &= \sum_{j=1}^{3} \int_{C_j} x\ dx + y\ dy \\ &= \int_0^1 (1)(0) + (t)(1)\ dt + \int_1^2 (-2t + 3)(-2) + (1)(0)\ dt \\ &\quad + \int_2^3 (-1)(0) + (-t + 3)(-1)\ dt \\ &= \left(\frac{t^2}{2}\right)\Big|_0^1 + \left(2t^2 - 6t\right)\Big|_1^2 + \left(\frac{t^2}{2} - 3t\right)\Big|_2^3 \\ &= \frac{1}{2} + 8 - 12 - 2 + 6 + \frac{9}{2} - 9 - \frac{4}{2} + 6 = 13 - 13 \\ &= 0 \end{aligned}$$

Now compute the double integral over region D

$$\begin{aligned} \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy &= \int_{-1}^{1} \int_{-1}^{1} (0 - 0)\ dx\ dy \\ &= \int_{-1}^{1} \int_{-1}^{1} 0\ dx\ dy \\ &= 0 \end{aligned}$$

Both answers are the same, so Green's theorem is satisfied. $\square$

3. Evaluate

$$\oint_C (2xy)\ dx + (xy^2)\ dy$$

where C is the closed, piecewise smooth curve formed by traveling in straight lines between the points $(-4,1), (-4,-3), (1,-2), (1,6)$, and back to $(-4,1)$, in that order.

***Solution*:** After drawing the points, you will see the region is a trapezoid, which has parallel vertical lines at $x=-4$ and $x=1$. The other two lines can be given as $y=\frac{1}{5}x-\frac{11}{5}$ and $y=x+5$. Therefore, we can describe the region D as

$$D = \left\{ (x,y) \middle| -4 \le x \le 1, \frac{1}{5}x - \frac{11}{5} \le y \le x+5 \right\}$$

Now we compute using Green's Theorem

$$\begin{aligned}
\oint_C M\ dx + N\ dy &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\ dx\ dy \\
&= \int_{-4}^{1} \int_{\frac{1}{5}x-\frac{11}{5}}^{x+5} (y^2 - 2x)\ dy\ dx \\
&= \int_{-4}^{1} \left(\frac{y^3}{3} - 2xy \right) \Bigg|_{y=\frac{1}{5}x-\frac{11}{5}}^{y=x+5} \\
&= \int_{-4}^{1} \left(\frac{1}{3}\left(\frac{1}{5}x - \frac{11}{5}\right)^3 - 2x\left(\frac{1}{5}x - \frac{11}{5}\right) - \frac{1}{3}(x+5)^3 + 2x(x+5) \right)\ dx \\
&= \frac{4}{375} \int_{-4}^{1} \left(31x^3 + 327x^2 + 903x + 4239\right)\ dx \\
&= \frac{4}{375} \left(\frac{31}{4}x^4 + \frac{327}{3}x^3 + \frac{903}{2}x^2 + 4239x \right) \Bigg|_{-4}^{1} \\
&= \frac{625}{3}
\end{aligned}$$

$\square$

4. Verify Green's theorem for the disc D with center $(0,0)$ and radius R and the following functions.

$$\begin{aligned}
&(a)\ M(x,y) = 5xy^2, \quad N(x,y) = -5x^2y \\
&(b)\ M(x,y) = x+y, \quad N(x,y) = y \\
&(c)\ M(x,y) = 6xy, \quad N(x,y) = 6xy \\
&(d)\ M(x,y) = 2y, \quad N(x,y) = x
\end{aligned}$$

Solution: For all of these questions, we must compute the line integral and the double integral over the region. Note since the path, and the region are the same, we define them first. Let the parameterization of the path C, the circle of radius R, be

$$\begin{aligned}
\mathbf{x}(t) &= (R\cos(t), R\sin(t)) && \text{for } t \in [0, 2\pi] \\
\mathbf{x}'(t) &= (-R\cos(t), R\sin(t)) && \text{for } t \in [0, 2\pi]
\end{aligned}$$

The region D is just the disk with center $(0,0)$ and radius R. It is easier to use polar coordinates, so the region is described in polar coordinates as

$$\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
dx\ dy &= r\ dr\ d\theta
\end{aligned}$$

for $r \in [0, R]$ and $\theta \in [0, 2\pi]$. We will convert all Cartesian integrals to polar.

(a) First compute the line integral

$$\begin{aligned}
\int_C M\ dx + N\ dy &= \int_C 5xy^2\ dx - 5x^2y\ dy \\
&= \int_0^{2\pi} \big((5R\cos(t)R^2\sin^2(t))(-R\sin(t))\big)\ dt \\
&\quad + \int_0^{2\pi} \big((-5R\sin(t)R^2\cos^2(t))(R\cos(t))\big)\ dt \\
&= -5R^4 \int_0^{2\pi} \big(\cos(t)\sin^3(t) + \cos^3(t)\sin(t)\big)\ dt \\
&= -5R^4 \left(\frac{1}{4}\sin^4(t) - \frac{1}{4}\cos^4(t)\right)\Bigg|_0^{2\pi} \\
&= 0
\end{aligned}$$

Now compute the double integral over region D.

$$\begin{aligned}
\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy &= \iint_D -10xy - 10xy\ dx\ dy \\
&= 20R^2 \int_0^{2\pi}\int_0^R \cos(\theta)\sin(\theta) r\ dr\ d\theta \\
&= 20R^2 \left(\int_0^R r\ dr\right)\left(\int_0^{2\pi} \cos(\theta)\sin(\theta)\ d\theta\right) \\
&= 20R^2 \left(\frac{r^2}{2}\Bigg|_0^R\right)\left(-\frac{1}{2}\cos^2(\theta)\Bigg|_0^{2\pi}\right) \\
&= 20R^2\left(\frac{R^2}{2}\right)(0) = 0
\end{aligned}$$

(b) First compute the line integral

$$\begin{aligned}
\int_C M\ dx + N\ dy &= \int_C (x+y)\ dx + y\ dy \\
&= \int_0^{2\pi} ((R\cos(t) + R\sin(t))(-R\sin(t)))\ dt \\
&\quad + \int_0^{2\pi} ((R\sin(t))(R\cos(t)))\ dt \\
&= -R^2 \int_0^{2\pi} \sin^2(t)\ dt \\
&= -\frac{R^2}{2}\left(t - \sin(t)\cos(t)\right)\Big|_0^{2\pi} \\
&= -\pi R^2
\end{aligned}$$

Now compute the double integral over region D.

$$\begin{aligned}
\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy &= \iint_D (0-1)\ dx\ dy \\
&= -\iint_D\ dx\ dy \\
&= -\pi R^2
\end{aligned}$$

where we interpret $\iint_D\ dx\ dy$ as the area of the region D. Since D is just a circle of radius R, its area is πR^2, and we avoid the integral computation.

(c) First compute the line integral

$$\begin{aligned}
\int_C M\ dx + N\ dy &= \int_C 6xy\ dx + 6xy\ dy \\
&= \int_0^{2\pi} (6(R\cos(t)R\sin(t))(-R\sin(t)))\ dt \\
&\quad + \int_0^{2\pi} (6(R\cos(t)R\sin(t))(R\cos(t)))\ dt \\
&= -6R^3 \int_0^{2\pi} (\cos(t)\sin^2(t) + \cos^2(t)\sin(t))\ dt \\
&= -6R^3 \left(\frac{1}{3}\sin^3(t) - \frac{1}{3}\cos^3(t)\right)\Big|_0^{2\pi} \\
&= 0
\end{aligned}$$

Now compute the double integral over region D.

$$\begin{aligned}
\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy &= \iint_D (6y - 6x)\ dx\ dy \\
&= 6\int_0^{2\pi}\int_0^R (R\sin(\theta) - R\cos(\theta))\, r\ dr\ d\theta \\
&= 6R\left(\int_0^R r\ dr\right)\left(\int_0^{2\pi} (\sin(\theta) - \cos(\theta))\ d\theta\right) \\
&= 6R\left(\frac{r^2}{2}\Big|_0^R\right)\left(-\cos(\theta) - \sin(\theta)|_0^{2\pi}\right) \\
&= 6R\left(\frac{R^2}{2}\right)(0) = 0
\end{aligned}$$

(d) First compute the line integral

$$\begin{aligned}
\int_C M\ dx + N\ dy &= \int_C 2y\ dx + x\ dy \\
&= \int_0^{2\pi} (2R\sin(t)(-R\sin(t)))\ dt \\
&\quad + \int_0^{2\pi} ((R\cos(t))(R\cos(t)))\ dt \\
&= R^2 \int_0^{2\pi} (-2\sin^2(t) + \cos(t))\ dt \\
&= R^2 \left(-\frac{1}{2}t + \frac{3}{2}\sin(t)\cos(t) \right)\Big|_0^{2\pi} \\
&= -\pi R^2
\end{aligned}$$

Now compute the double integral over region D.

$$\begin{aligned}
\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\ dx\ dy &= \iint_D (1-2)\ dx\ dy \\
&= -\iint_D\ dx\ dy \\
&= -\pi R^2
\end{aligned}$$

where we interpret $\displaystyle\iint_D\ dx\ dy$ as the area of the region D. Since D is just a circle of radius R, its area is πR^2, and we avoid the integral computation.

$\square$

5. Evaluate

$$\oint_C (2x^3 - y^3)\ dx + (4x^3 + y^3)\ dy$$

where C is the unit circle.

Solution: Since C is the unit circle, we can describe the region D as

$$D = \left\{(x, y) \middle| -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\right\}$$

Now we compute using Green's Theorem

$$\begin{aligned}
\oint_C M\ dx + N\ dy &= \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\ dx\ dy \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (12x^2 + 3y^2)\ dy\ dx \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (9x^2 + 3x^2 + 3y^2)\ dy\ dx \\
&= \int_{0}^{1} \int_{0}^{2\pi} r(9r^2 \cos^2(\theta) + 3r^2)\ d\theta\ dr \\
&= \int_{0}^{1} \int_{0}^{2\pi} (9r^3 \cos^2(\theta) + 3r^3)\ d\theta\ dr \\
&= \left(\int_0^1 3r^3\ dr\right) \left(\int_0^{2\pi} (3\cos^2(\theta) + 1)\ d\theta\right) \\
&= \left(\frac{3}{4} r^4 \Big|_0^1\right) \left(\left(\frac{5}{2}\theta + \frac{3}{4}\sin(2\theta)\right)\Big|_0^{2\pi}\right) \\
&= \left(\frac{3}{4}\right)\left(\frac{10\pi}{2}\right) \\
&= \frac{15\pi}{4}
\end{aligned}$$

□

6. Find the area bounded by one arc of the cycloid $x = a(\theta - \sin(\theta))$, $y = a(1 - \cos(\theta))$, where $a > 0$, and $0 \le \theta \le 2\pi$, and the x-axis using Green's theorem.

$$\oint_C (2x^3 - y^3)\ dx + (4x^3 + y^3)\ dy$$

where C is the unit circle.

Solution: Note that the area of a region D is given by $\iint_D 1\ dA$. To use Green's Theorem, we need to construct a vector field $\mathbf{F} = (M, N)$, such that

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = f(x, y) = 1$$

There is no unique choice of $\mathbf{F}$, so we just choose one that works, say

$$\mathbf{F} = \left(-\frac{y}{2}, \frac{x}{2}\right)$$

We can compute the left hand side of Green's Theorem using $\mathbf{x}(\theta) = (a(\theta - \sin(\theta)), a(1 - \cos(\theta)))$ to get

$$\begin{aligned}
\oint_C M\ dx + N\ dy &= \frac{1}{2}\oint_C -y\ dx + x\ dy \\
&= \frac{1}{2}\int_{2\pi}^{0} (-a(1 - \cos(\theta))(a(1 - \cos(\theta))) + (a(\theta - \sin(\theta)))(a\sin(\theta)))\ d\theta \\
&= \frac{1}{2}\int_{2\pi}^{0} (-a^2(1 - \cos(\theta))^2 + a^2(\theta\sin(\theta) - \sin^2(\theta)))\ d\theta \\
&= \frac{1}{2}a^2\int_{2\pi}^{0} (-(1 - \cos(\theta))^2 + (\theta\sin(\theta) - \sin^2(\theta)))\ d\theta \\
&= \frac{1}{2}a^2\int_{2\pi}^{0} (-1 + 2\cos(\theta) - \cos^2(\theta) + \theta\sin(\theta) - \sin^2(\theta))\ d\theta \\
&= \frac{1}{2}a^2\int_{2\pi}^{0} (-2 + 2\cos(\theta) + \theta\sin(\theta))\ d\theta \\
&= \frac{1}{2}a^2\left(-2\theta + 3\sin(\theta) - \theta\cos(\theta)\right)\Big|_{2\pi}^{0} \\
&= \frac{1}{2}a^2(6\pi) \\
&= 3\pi a^2
\end{aligned}$$

where the bounds are chosen backwards due to the orientation of the curve, and get a positive area.

$\square$

7. Evaluate

$$\oint_C y^2 \; dx + x^2 \; dy$$

for C the boundary of the region that lies between the curves $y = x$ and $y = x^2$.

***Solution*:** Note that the region D has the bounds $0 \leq x \leq 1$ and $x^2 \leq y \leq x$. We also let $M = y^2$, and $N = x^2$. Then by Green's Theorem

$$\begin{aligned}
\oint_C y^2 \; dx + x^2 \; dy &= \iint_D (2x - 2y) \; dy \; dx \\
&= \int_0^1 \int_{x^2}^{x} (2x - 2y) \; dy \; dx \\
&= \int_0^1 \left. (2xy - y^2) \right|_{x^2}^{x} \; dx \\
&= \int_0^1 \left(x^2 - 2x^3 + x^4 \right) \; dx \\
&= \left. \frac{x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5} \right|_0^1 \\
&= \frac{1}{30}
\end{aligned}$$

□

8. Evaluate

$$\oint_C e^x \sin(y) \; dx + e^x \cos(y) \; dy$$

where C is the closed curve consisting of the semicircle $y = \sqrt{1 - x^2}$ and the interval $-1 \leq x \leq 1$.

***Solution*:** Note that the region D has the bounds $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1 - x^2}$. We also let $M = e^x \sin(y)$, and $N = e^x \cos(y)$. Then by Green's Theorem

$$\begin{aligned}
\oint_C e^x \sin(y) \; dx + e^x \cos(y) \; dy &= \iint_D (e^x \cos(y) - e^x \cos(y)) \; dy \; dx \\
&= \int_{-1}^{1} \int_0^{\sqrt{1-x^2}} 0 \; dy \; dx \\
&= 0
\end{aligned}$$

□

9. Evaluate

$$\oint_C (y + e^{\sqrt{x}})\ dx + (2x + \cos(y^2))\ dy$$

where C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

Solution: Note that the region D has the bounds $0 \le x \le 1$ and $x^2 \le y \le \sqrt{x}$. We also let $M = y + e^{\sqrt{x}}$, and $N = 2x + \cos(y^2)$. Then by Green's Theorem

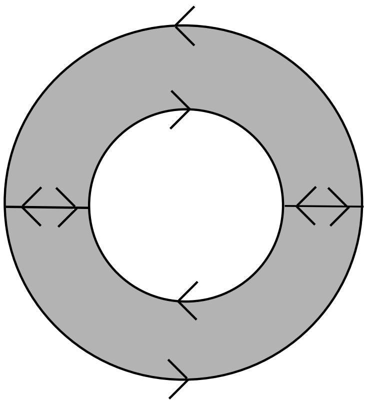
$$\begin{aligned}\oint_C (y + e^{\sqrt{x}})\ dx + (2x + \cos(y^2))\ dy &= \iint_D (2-1)\ dy\ dx \\ &= \int_0^1 \int_{x^2}^{\sqrt{x}} dy\ dx \\ &= \int_0^1 (\sqrt{x} - x^2)\ dx \\ &= \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Bigg|_0^1 = \frac{1}{3}\end{aligned}$$

□

10. Evaluate

$$\oint_C (1 - y^3)\ dx + (x^3 + e^{y^2})\ dy$$

where C is the boundary of the region between the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$, positively oriented with respect to the outer circle.



Solution: The picture of the situation is given above. Notice that there are many C_i that would have to be considered to do the path integral separately. This is a hint to do the problem using Green's Theorem. We will hold off on the boundaries of the region D for now, but let $M = 1 - y^3$, and $N = x^3 + e^{y^2}$. Then by Green's Theorem

$$\begin{aligned}\oint_C (1 - y^3)\ dx + (x^3 + e^{y^2})\ dy &= \iint_D (3x^2 - (-3y^2))\ dy\ dx \\ &= 3\iint_D (x^2 + y^2)\ dy\ dx\end{aligned}$$

Now notice that the problem will be very easy to do using polar coordinates, since the shaded region is bounded by $2 \leq r \leq 3$ and $0 \leq \theta \leq 2\pi$. Therefore, we have

$$\begin{aligned}
&= 3 \iint_D (x^2 + y^2)\ dy\ dx \\
&= 3 \int_0^{2\pi} \int_2^3 r^3\ dr\ d\theta \\
&= \frac{3\pi}{2} r^4 \bigg|_2^3 \\
&= \frac{195}{2}\pi
\end{aligned}$$

$\square$