MATH 010B - Spring 2018

Worked Problems - Section 8.2

For these questions, recall what Stokes' Theorem states:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot \, d\mathbf{S}$$
$$\iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S} = \iint_D (\nabla \times \mathbf{F})\left(\mathbf{T}(u,v)\right) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \; du \; dv = \int_{\partial S} \mathbf{F} \cdot \, d\mathbf{S}$$

which allows us to compute either a surface integral of the curl of the vector field $\mathbf{F}$, or the line integral over the boundary of the surface, ∂S. The second formula allows us to compute the surface integral using a parameterization $\mathbf{T}(u, v)$. Finally, we may use the following helpful formula

$$\iint_S \mathbf{G} \cdot \, d\mathbf{S} = \iint_D \left(G_1 \left(-\frac{\partial z}{\partial x} \right) + G_2 \left(-\frac{\partial z}{\partial y} \right) + G_3 \right) \; dx \; dy$$

for some graph $z = f(x, y)$.

1. Let S be the portion of the plane $2x+3y+z = 6$ lying between the points $(-1, 1, 5)$, $(2, 1, -1)$, $(2, 3, -7)$, and $(-1, 3, -1)$. Find parameterizations for both the surface S and its boundary ∂S. Be sure that their respective orientations are compatible with Stokes' theorem.

***Solution*:** Just construct linear pieces for each component using a table.

For the line from $(-1, 1, 5)$ to $(2, 1, -1)$, and for $t \in [0, 1]$, use the table

t	x(t)	y(t)	z(t)
0	-1	1	5
1	2	1	-1

which gives us the first component as $\mathbf{S}_1(t) = (x(t), y(t), z(t)) = (3t-1, 1, -6t+5)$. We do the same for $(2, 1, -1)$ to $(2, 3, -7)$, and for $t \in [1, 2]$

t	x(t)	y(t)	z(t)
1	2	1	-1
2	2	3	-7

which gives us the second component as $\mathbf{S}_2(t) = (x(t), y(t), z(t)) = (2, 2t-1, -6t+5)$. We do the same for $(2, 3, -7)$ to $(-1, 3, -1)$, and for $t \in [2, 3]$

t	x(t)	y(t)	z(t)
2	2	3	-7
3	-1	3	-1

which gives us the third component as $\mathbf{S}_3(t) = (x(t), y(t), z(t)) = (-3t+8, 3, 6t-19)$. We do the same for $(-1, 3, -1)$ to $(-1, 1, 5)$, and for $t \in [3, 4]$

t	x(t)	y(t)	z(t)
3	-1	3	-1
4	-1	1	5

which gives us the fourth component as $\mathbf{S}_4(t) = (x(t), y(t), z(t)) = (-1, -2t+9, 6t-19)$. Therefore, the boundary ∂S can be given as the piecewise curve defined as

$$\mathbf{f}(t) = \begin{cases} \mathbf{S}_1(t) = (3t-1, 1, -6t+5) & \text{for } t \in [0,1] \\ \mathbf{S}_2(t) = (2, 2t-1, -6t+5) & \text{for } t \in [1,2] \\ \mathbf{S}_3(t) = (-3t+8, 3, 6t-19) & \text{for } t \in [2,3] \\ \mathbf{S}_4(t) = (-3t+8, 3, 6t-19) & \text{for } t \in [3,4] \end{cases}$$

Now the S is the portion of the plane $2x + 3y + z = 6$ lying between the points given. Note that we can write the surface as $z = 6 - 2x - 3y$. The most natural parametrization to choose would be to let $x = u$ and $y = v$, where $x = u \in [-1, 2]$ and $y = v \in [1, 3]$. The bounds come from looking at the range of the x and y coordinates of the points, which corresponds to the projection of the region in the xy-plane. Plugging this into the equation for z, we get the parameterization of the surface as

$$\mathbf{T}(u, v) = (u, v, 6 - 2u - 3v)$$

$\square$

2. Verify Stokes' theorem for the given surface S and boundary ∂S, and vector field $\mathbf{F}$.

$$\begin{aligned} S &= \{(x, y, z) \mid z = 1 - x^2 - y^2, z \geq 0\}, \\ \partial S &= \{(x, y) \mid x^2 + y^2 = 1\} \\ \mathbf{F} &= (z, x, 5zx + 4xy) \end{aligned}$$

Solution: Note that the surface S is a paraboloid that opens down, with ∂S being the base of the surface on the $z = 0$ plane, which is a circle of radius 1. First we compute the left hand side of Stokes' theorem. We compute the cross product first

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & x & 5zx + 4xy \end{vmatrix} = (4x, -(5z - 4y - 1), 1)$$

Then we use the formula from page 1, where we let $\mathbf{G} = (4x, -(5z - 4y - 1), 1)$, and D is the unit disk. Thus

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S} &= \iint_S \mathbf{G} \cdot \, d\mathbf{S} \\ &= \iint_D \left(G_1 \left(-\frac{\partial z}{\partial x} \right) + G_2 \left(-\frac{\partial z}{\partial y} \right) + G_3 \right) \, dx \, dy \\ &= \iint_D ((4x)(2x) - (5z - 4y - 1)(2y) + 1) \, dx \, dy \\ &= \iint_D (8x^2 - 10yz - 8y^2 + 2y + 1) \, dx \, dy \\ &= \iint_D (8x^2 - 8y^2 + 2y + 1) \, dx \, dy \qquad z = 0 \text{ on the disk } D \\ &= \int_0^{2\pi} \int_0^1 (8\cos^2(\theta) - 8\sin^2(\theta) + 2\sin(\theta) + 1) r \, dr \, d\theta \\ &\text{use the identity} \quad \cos^2(t) - \sin^2(t) = \cos(2t) \\ &= \left(\int_0^1 r \, dr \right) \left(\int_0^{2\pi} (8\cos(2\theta) + 2\sin(\theta) + 1) \, d\theta \right) \\ &= \left(\frac{r^2}{2} \bigg|_0^1 \right) \left(t + (8\sin(\theta) - 2)\cos(\theta)|_0^{2\pi} \right) = \frac{1}{2}(2\pi) = \pi \end{aligned}$$

For the line integral, we need the parameterization for the unit circle on the plane $z = 0$, thus

$$\begin{aligned} \mathbf{x}(t) &= (\cos(\theta), \sin(\theta), 0) \quad \text{for } t \in [0, 2\pi] \\ \mathbf{x}'(t) &= (-\sin(\theta), \cos(\theta), 0) \quad \text{for } t \in [0, 2\pi] \end{aligned}$$

Then we have that

$$\mathbf{F}(\mathbf{x}(t)) = (0, \cos(t), 4\sin(t)\cos(t))$$

Using the line integral formula

$$\begin{aligned}
\int_{\partial S} \mathbf{F} \cdot \ d\mathbf{S} &= \int_0^{2\pi} \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \ dt \\
&= \int_0^{2\pi} (0, \cos(t), 4\sin(t)\cos(t)) \cdot (-\sin(\theta), \cos(\theta), 0) \ dt \\
&= \int_0^{2\pi} \cos^2(t) \ dt \\
&= \left(\frac{1}{2}(t + \sin(t)\cos(t)) \right) \Big|_0^{2\pi} \\
&= \frac{1}{2}(2\pi) \\
&= \pi
\end{aligned}$$

Both answers are the same, so Stokes' theorem is satisfied. $\square$

3. Let $\mathbf{F} = (x^2, 5xy + x, z)$. Let C be the circle $x^2 + y^2 = 81$, and S be the disk $x^2 + y^2 \leq 81$ within the plane $z = 0$.

$$(a) \text{ Determine } \iint_S \mathbf{F} \cdot \, d\mathbf{S}$$

$$(b) \text{ Determine } \iint_C \mathbf{F} \cdot \, d\mathbf{S}$$

$$(c) \text{ Determine } \iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S}$$

Note that (b) and (c) are the two sides of Stokes' theorem.

Solution: (a) Recall that the formula for integrals for vector fields over surfaces is given by

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{T}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \; du \; dv$$

Now we must compute all of the pieces. We parameterize the disk on the plane $z = 0$ by

$$\begin{aligned}
\mathbf{T}(u,v) &= (u\cos(v), u\sin(v), 0) \\
\mathbf{T}_u &= (\cos(v), \sin(v), 0) \\
\mathbf{T}_v &= (-u\sin(v), u\cos(v), 0) \\
\mathbf{T}_u \times \mathbf{T}_v &= (0, 0, u) \\
\mathbf{F}(\mathbf{T}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) &= (u^2\cos^2(v), 5u^2\cos(v)\sin(v) + u\cos(v), 0) \cdot (0, 0, u) = 0
\end{aligned}$$

for $u \in [0, 9]$ and $v \in [0, 2\pi]$. So then the problem becomes

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D 0 \; du \; dv = 0$$

(b) Recall that the formula for integrals for vector fields over paths is given by

$$\int_C \mathbf{F} \cdot d\mathbf{S} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot (\mathbf{x}'(t)) \; dt$$

Now we must compute all of the pieces. We parameterize the circle on the plane $z = 0$ by

$$\begin{aligned}
\mathbf{x}(t) &= (9\cos(t), 9\sin(t), 0) \\
\mathbf{x}'(t) &= (-9\sin(t), 9\cos(t), 0) \\
\mathbf{F}(\mathbf{x}(t)) \cdot (\mathbf{x}'(t)) &= (\cos^2(t), 405\cos(t)\sin(t) + 9\cos(t), 0) \cdot (-9\sin(t), 9\cos(t), 0) \\
&= 3636\cos^2(t)\sin(t) + 81\cos^2(t)
\end{aligned}$$

for $t \in [0, 2\pi]$. So then the problem becomes

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{S} &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot (\mathbf{x}'(t)) \; dt \\
&= \int_0^{2\pi} (3636\cos^2(t)\sin(t) + 81\cos^2(t)) \; dt \\
&= \left(-1212\cos^3(t) + \frac{81}{2}(t + \sin(t)\cos(t)) \right)\Big|_0^{2\pi} \\
&= 81\pi
\end{aligned}$$

(c) Compute the cross product first

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & 5xy + x & z \end{vmatrix} = (0, 0, 5y + 1)$$

Then we use the formula from page 1, the parameterization from (a) (plug parameterization into the curl), and the region D as the disk, we have

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S} &= \iint_D (\nabla \times \mathbf{F}) (\mathbf{T}(u, v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) \, du \, dv \\
&= \iint_D (0, 0, 5u \sin(v) + 1) \cdot (0, 0, u) \, du \, dv \\
&= \int_0^{2\pi} \int_0^9 (5u^2 \sin(v) + u) \, du \, dv \\
&= \int_0^{2\pi} \left(5\frac{9^3}{3} \sin(v) + \frac{81}{2} \right) \, dv \\
&= \left(-5\frac{9^3}{3} \cos(v) + \frac{81}{2} v \right) \Bigg|_0^{2\pi} \\
&= \frac{81}{2}(2\pi) = 81\pi
\end{aligned}$$

□

4. Let C be the closed, piecewise smooth curve formed by traveling in straight lines between the points $(0,0,0), (2,0,18), (3,2,27), (1,2,9)$, and back to the origin, in that order. (Thus the surface S lying interior to C is contained in the plane $z = 9x$.) Use Stokes' theorem to evaluate the following integral.

$$\int_C (z\cos(x))\ dx + (x^2yz)\ dy + (yz)\ dz$$

***Solution*:** Note that we can reconstruct the vector field $\mathbf{F}$ by taking the pieces inside the integral, thus $\mathbf{F} = (z\cos(x), x^2yz, yz)$. Compute the cross product first

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z\cos(x) & x^2yz & yz \end{vmatrix} = (z - x^2y, \cos(x), 2xyz)$$

Now we need the parameterization of the region described in the problem. Note that the region lies in the plane $z = 9x$, thus if we take $x = u$, $y = v$, and $z = 9u$, we have

$$\begin{aligned} \mathbf{T}(u,v) &= (u, v, 9u) \\ \mathbf{T}_u &= (1,0,9) \\ \mathbf{T}_v &= (0,1,0) \\ \mathbf{T}_u \times \mathbf{T}_v &= (-9,0,1) \\ \mathbf{F}(\mathbf{T}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v) &= (9u - u^2v, \cos(u), 18u^2v) \cdot (-9,0,1) \\ &= (27u^2v - 81u) \end{aligned}$$

for $u \in \left[\frac{v}{2}, \frac{v+4}{2}\right]$ and $v \in [0,2]$. To determine those bounds, take the projection of the surface into the xy-plane, and you will see it is a parallelogram which is bounded by the lines $y = 0$, $y = 2$, $y = 2x$, and $y = 2x - 4$. This then gives the bounds when we use u and v.

Now we use the formula from page 1, the parameterization above, and the region D as the parallelogram, we have

$$\begin{aligned} \iint_S (\nabla \times \mathbf{F}) \cdot\ d\mathbf{S} &= \iint_D (\nabla \times \mathbf{F})\,(\mathbf{T}(u,v)) \cdot (\mathbf{T}_u \times \mathbf{T}_v)\ du\ dv \\ &= \iint_D (27u^2v - 81u)\ du\ dv \\ &= \int_0^2 \int_{\frac{v}{2}}^{\frac{v+4}{2}} (27u^2v - 81u)\ du\ dv \\ &= \int_0^2 \left(9u^3v - 81uv\right)\Big|_{u=\frac{v}{2}}^{u=\frac{v+4}{2}}\ dv \\ &= -144 \end{aligned}$$

$\square$