MATH 010B - Spring 2018

Worked Problems - Section 8.3

For some of these questions, we will use the following corollary from the book:

Corollary 0.1 *If $\boldsymbol{F}$ is a C^1 vector field on $\mathbb{R}^2$ defined as $\boldsymbol{F} = (M(x,y), N(x,y))$, and $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, then $\boldsymbol{F} = \nabla f$ for some function f on $\mathbb{R}^2$.*

1. Determine which of the following vector fields $\mathbf{F}$ in the plane is the gradient of a scalar function f. If such an f exists, find it. If an answer does not exist, state that is does not exist (DNE).

$$(a)\ \mathbf{F}(x,y) = (5x, 5y)$$
$$(b)\ \mathbf{F}(x,y) = (3xy, 3xy)$$
$$(c)\ \mathbf{F}(x,y) = (4x^2 + 4y^2, 8xy)$$

Solution: (a) First we check the condition in the Corollary above to see if the function f exists

$$\frac{\partial M}{\partial y} = 0$$
$$\frac{\partial N}{\partial x} = 0$$

Therefore, the hypothesis of the Corollary is satisfied. Now we find the f. For the f to exist, we must require that

$$\frac{\partial f}{\partial x} = 5x \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 5y$$

So we integrate the first expression above to get that

$$\begin{aligned}
\frac{\partial f}{\partial x} = 5x \quad &\Rightarrow \quad f(x,y) = \frac{5x^2}{2} + h(y) \\
&\Rightarrow \quad \frac{\partial f}{\partial y} = h'(y) = 5y \\
&\Rightarrow \quad h(y) = \frac{5y^2}{2} + C \\
&\Rightarrow \quad f(x,y) = \frac{5x^2}{2} + \frac{5y^2}{2} + C
\end{aligned}$$

(b) First we check the condition in the Corollary above to see if the function f exists

$$\frac{\partial M}{\partial y} = 3x$$
$$\frac{\partial N}{\partial x} = 3y$$

Therefore, the hypothesis of the Corollary is not satisfied, so the f does not exist.

(c) First we check the condition in the Corollary above to see if the function f exists

$$\frac{\partial M}{\partial y} = 8y$$
$$\frac{\partial N}{\partial x} = 8y$$

Therefore, the hypothesis of the Corollary is satisfied. Now we find the f. For the f to exist, we must require that

$$\frac{\partial f}{\partial y} = 8xy \qquad \text{and} \qquad \frac{\partial f}{\partial x} = 4x^2 + 4y^2$$

So we integrate the first expression above to get that

$$\begin{aligned}
\frac{\partial f}{\partial y} = 8xy \quad &\Rightarrow \quad f(x,y) = 4xy^2 + h(x) \\
&\Rightarrow \quad \frac{\partial f}{\partial y} = 4y^2 + h'(x) = 4x^2 + 4y^2 \\
&\Rightarrow \quad h'(x) = 4x^2 \\
&\Rightarrow \quad h(x) = \frac{4}{3}x^3 + C \\
&\Rightarrow \quad f(x,y) = 4xy^2 + \frac{4}{3}x^3 + C
\end{aligned}$$

□

2. Let $\mathbf{F}(x,y,z) = (6e^x \sin(y), 3x^2z, 3x^2y)$. Find a function f such that $\mathbf{F} = \nabla f$.

Solution: We proceed as in number one, except now we have 3 variables. Now, note that

$$
\begin{aligned}
\frac{\partial f}{\partial y} = 3x^2z \quad &\Rightarrow \quad f(x,y,z) = 3x^2yz + g(x) + h(z) \\
&\Rightarrow \quad \frac{\partial f}{\partial z} = 3x^2y + h'(z) = 3x^2y \\
&\Rightarrow \quad h'(z) = 0 \\
&\Rightarrow \quad h(z) = C_1 \\
&\Rightarrow \quad \frac{\partial f}{\partial x} = 6xyz + g'(x) = 6xyz + 3\sin(x) \\
&\Rightarrow \quad g'(x) = 3\sin(x) \\
&\Rightarrow \quad g(x) = -3\cos(x) + C_2 \\
&\Rightarrow \quad f(x,y,z) = 3x^2yz - 3\cos(x) + C_2 + C_1 \\
&\Rightarrow \quad f(x,y,z) = 3x^2yz - 3\cos(x) + C
\end{aligned}
$$

□

3. Let $\mathbf{F}(x,y,z) = (6e^x \sin(y), 6e^x \cos(y), 4z^2)$. Evaluate $\int_C \mathbf{F} \cdot \, d\mathbf{s}$, where $\mathbf{c}(t) = (8\sqrt{t}, t^3, e^{\sqrt{t}})$ for $t \in [0,1]$.

Solution: First we will establish that the vector field is conservative by showing that $\nabla \times \mathbf{F} = \mathbf{0}$. If the vector field is conservative, we can choose any path to integrate along, as long as the endpoints are the same. So, we have

$$
\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 6e^x \sin(y) & 6e^x \cos(y) & 4z^2 \end{vmatrix} = (0, 0, 6e^x\cos(y) - 6e^x\cos(y)) = (0,0,0)
$$

thus the vector field is conservative. We choose the simplest possible path to replace $\mathbf{c}(t)$. Note that $\mathbf{c}(t)$ starts at $(0,0,1)$ at $t=0$, and ends at $(8,1,e)$ at $t=1$. Thus we choose a straight line between the points, which you can use the table method to determine is $\mathbf{x}(t) = (8t, t, e^t)$, with $\mathbf{x}'(t) = (8, 1, e^t)$ for $t \in [0,1]$. Thus we have

$$
\begin{aligned}
\int_C \mathbf{F} \cdot \, d\mathbf{s} &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \, dt \\
&= \int_0^1 (6e^{8t}\sin(t), 6e^{8t}\cos(t), 4e^{2t}) \cdot (8, 1, e^t) \, dt \\
&= \int_0^1 48e^{8t}\sin(t) + 6e^{8t}\cos(t) + 4e^{3t} \, dt \\
&= \left(\frac{4}{3}e^{3t} + 6e^{8t}\sin(t) \right) \Bigg|_0^1 \\
&= \frac{4}{3}e^3 + 6e^8\sin(1) - \frac{4}{3}
\end{aligned}
$$

□

4. Determine if the following vector fields $\mathbf{F}$ are gradient fields. In other words, is there a scalar function f such that $\mathbf{F} = \nabla f$. If such an f exists, find it. If an answer does not exist, state that is does not exist (DNE).

$$\begin{aligned}
&(a)\ \mathbf{F}(x,y) = (2x + 3y^2 - y\sin(x), 6xyz + \cos(x))\\
&(b)\ \mathbf{F}(x,y,z) = (8x^2z^2, 7x^2y^2, 5y^2z^2)\\
&(c)\ \mathbf{F}(x,y) = (y^3 + 8, 3xy^2 + 4)\\
&(d)\ \mathbf{F}(x,y,z) = \left(xe^{x^2+y^2} + 4xy, ye^{x^2+y^2} + 8y^3z, 2y^4\right)
\end{aligned}$$

Solution: (a) Note that this question is a function of two variables, x and y, but there is a z present. First we check the condition in the Corollary above to see if the function f exists

$$\frac{\partial M}{\partial y} = 6y - \sin(x)$$
$$\frac{\partial N}{\partial x} = 6yz - \sin(x)$$

Therefore, the hypothesis of the Corollary is not satisfied, so the f does not exist.

(b) An equivalent condition to the Corollary above is to check that $\nabla \times \mathbf{F} = \mathbf{0}$. We compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 8x^2z^2 & 7x^2y^2 & 5y^2z^2 \end{vmatrix} = (10yz^2, -(0 - 16x^2z), 14xy^2) = (10yz^2, 16x^2z, 14xy^2)$$

which is not equal to $(0,0,0)$. Therefore, the f does not exist.

(c) First we check the condition in the Corollary above to see if the function f exists

$$\frac{\partial M}{\partial y} = 3y^2$$
$$\frac{\partial N}{\partial x} = 3y^2$$

Therefore, the hypothesis of the Corollary is satisfied, so we need to determine f. So we need

$$\frac{\partial f}{\partial x} = y^3 + 8 \qquad \text{and} \qquad \frac{\partial f}{\partial y} = 3xy^2 + 4$$

So we integrate the first expression above to get that

$$\begin{aligned}
\frac{\partial f}{\partial x} = y^3 + 8 \quad &\Rightarrow \quad f(x,y) = xy^3 + 8x + h(y)\\
&\Rightarrow \quad \frac{\partial f}{\partial y} = 3xy^2 + h'(y) = 3xy^2 + 4\\
&\Rightarrow \quad h'(y) = 4\\
&\Rightarrow \quad h(y) = 4y + C\\
&\Rightarrow \quad f(x,y) = xy^3 + 8x + 4y + C
\end{aligned}$$

(d) An equivalent condition to the Corollary above is to check that $\nabla \times \mathbf{F} = \mathbf{0}$. We compute

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xe^{x^2+y^2} + 4xy & ye^{x^2+y^2} + 8y^3 z & 2y^4 \end{vmatrix} = (8y^3-8y^3, 0, 2xye^{x^2+y^2}-2xye^{x^2+y^2}-4x)$$

which gives $(0,0,-4x) \neq (0,0,0)$. Therefore, the f does not exist. $\square$

5. Evaluate the line integral

$$\int_C 2xyz\ dx + x^2 z\ dy + x^2 y\ dz$$

where C is an oriented simple closed curve connecting $(1,1,1)$ and $(1,2,4)$.

***Solution*:** We compute the value of $\nabla \times \mathbf{F} = \mathbf{0}$ first, since if we show that the vector field is conservative, we can choose any path to compute the line integral. We construct $\mathbf{F}$ from the problem statement, so that $\mathbf{F} = (2xyz, x^2 z, x^2 y)$. We compute

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xyz & x^2 z & x^2 y \end{vmatrix} = (x^2 - x^2, -(2xy - 2xy), 2xz - 2xz) = (0,0,0)$$

which is not equal to $(0,0,0)$. Therefore, the $\mathbf{F}$ is a conservative vector field, so we can choose any path we want. We simply take the line that connects $(1,1,1)$ and $(1,2,4)$. Using the table method, we can determine that the line is parameterized as $\mathbf{x}(t) = (1, t+1, 3t+1)$, with $\mathbf{x}'(t) = (0,1,3)$ for $t \in [0,1]$. Now we compute the line integral

$$\begin{aligned} \int_C 2xyz\ dx + x^2 z\ dy + x^2 y\ dz &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t)\ dt \\ &= \int_0^1 (2(1)(t+1)(3t+1), (1)^2(3t+1), (1)^2(t+1)) \cdot (0,1,3)\ dt \\ &= \int_0^1 6t + 4\ dt \\ &= \left(3t^2 + 4t\right)\Big|_0^1 \\ &= 7 \end{aligned}$$

$\square$

6. Suppose

$$\nabla f(x,y,z) = \left(2xyze^{x^2}, ze^{x^2}, ye^{x^2}\right)$$

If $f(0,0,0) = 1$, find $f(2,2,3)$.

***Solution*:** We have seen this problem once before, and we approached it in a similar manner as problem 1. Here we will use the Fundamental Theorem for Line Integrals. The theorem states

Theorem 0.1 *Suppose that C is a smooth curve given by $\boldsymbol{x}(t)$ for $t \in [a,b]$, and that f is a function whose gradient ∇f is continuous on the curve C. Then,*

$$\int_C (\nabla f) \cdot d\boldsymbol{s} = f(\boldsymbol{x}(b)) - f(\boldsymbol{x}(a))$$

First, we define the curve $\mathbf{x}(t) = (2t, 2t, 3t)$, with $\mathbf{x}'(t) = (2,2,3)$ for $t \in [0,1]$. This is a straight line from point to point. Just use the table method to determine the parameterization. The curve is smooth, as it is a line, and the gradient is smooth, as it is polynomials and exponentials. Therefore, we can apply the theorem. So we can compute the left hand side of this expression, since we know that $f(\mathbf{x}(a)) = f(0,0,0) = 1$, and we want to figure out what $f(\mathbf{x}(b)) = f(2,2,3)$ is. First compute the inside of the integral

$$\begin{aligned}
(\nabla f)(\mathbf{x}(t)) \cdot \mathbf{x}'(t) &= \left(2(2t)(2t)(3t)e^{(2t)^2}, 3te^{(2t)^2}, 2te^{(2t)^2}\right) \cdot (2,2,3) \\
&= \left(24t^3e^{4t^2}, 3te^{4t^2}, 2te^{4t^2}\right) \cdot (2,2,3) \\
&= 48t^3e^{4t^2} + 12te^{4t^2}
\end{aligned}$$

where the notation means the composition of the gradient and the parameterization, not multiplication. Now we compute the line integral

$$\begin{aligned}
\int_C (\nabla f) \cdot \; d\mathbf{s} &= \int_a^b (\nabla f)(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \; dt \\
&= \int_0^1 48t^3e^{4t^2} + 12te^{4t^2} \; dt \\
&= \left(6t^2e^{4t^2}\right)\Big|_0^1 \\
&= 6e^4
\end{aligned}$$

Thus we have that

$$\begin{aligned}
\int_C (\nabla f) \cdot d\mathbf{s} &= f(\mathbf{x}(b)) - f(\mathbf{x}(a)) \\
6e^4 &= f(\mathbf{x}(b)) - 1 \\
f(\mathbf{x}(b)) &= 6e^4 + 1
\end{aligned}$$

so we get the solution $f(\mathbf{x}(b)) = f(2,2,3) = 6e^4 + 1$. $\square$