MATH 010B - Spring 2018

Worked Problems - Section 8.4

Recall the Divergence theorem

$$\iiint_W (\nabla \cdot \mathbf{F})\ dV = \iint_{\partial W} \mathbf{F} \cdot\ d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

which states we can compute either a volume integral of the divergence of $\mathbf{F}$, or the surface integral over the boundary of the region W, or the surface integral with normal $\mathbf{n}$. We compute whichever one is the easiest to do, as they are equivalent by the theorem.

1. Verify the Divergence theorem for the given region W, boundary ∂W oriented outward, and the vector field $\mathbf{F}$.

$$W = [0,1] \times [0,1] \times [0,1] \qquad \mathbf{F}(2x, 3y, 2z)$$

***Solution*:** First we will compute the volume integral side of the Divergence theorem. Computing the divergence we have

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (2x, 3y, 2z) \\ &= 2 + 3 + 2 \\ &= 7\end{aligned}$$

Computing the volume integral, we have

$$\begin{aligned}\iiint_W (\nabla \cdot \mathbf{F})\ dV &= \iiint_W 7\ dV \\ &= 7 \iiint_W\ dV \\ &= 7(1)(1)(1) \\ &= 7\end{aligned}$$

where we interpret the volume integral $\iiint_W\ dV$ as the volume of the box W which is the product of the length, width and height of the box.

Now for the surface integral side, we can use the description in the book for the unit cube. We cannot describe the surface of the cube in one equation, therefore, we have the bottom, top, back, front, left, and right of the box given by planes

$$\begin{aligned}S_1 &: \quad z = 0, \quad x, y \in [0,1] \\ S_2 &: \quad z = 1, \quad x, y \in [0,1] \\ S_3 &: \quad x = 0, \quad y, z \in [0,1] \\ S_4 &: \quad x = 1, \quad y, z \in [0,1] \\ S_5 &: \quad y = 0, \quad x, z \in [0,1] \\ S_6 &: \quad y = 1, \quad x, z \in [0,1]\end{aligned}$$

and the normal for each face is given as

$$\begin{aligned}
\mathbf{n}_2 = -\mathbf{n}_1 = \mathbf{k} = (0,0,1) \\
\mathbf{n}_4 = -\mathbf{n}_3 = \mathbf{i} = (1,0,0) \\
\mathbf{n}_6 = -\mathbf{n}_5 = \mathbf{j} = (0,1,0)
\end{aligned}$$

We simplify the surface integral by using the third formula at the top of page 1, where we dot the normal vectors $\mathbf{n}_i$ with the vector field $\mathbf{F} = (F_1, F_2, F_3)$ to get

$$\begin{aligned}
\iint_{\partial W} \mathbf{F} \cdot \, d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \\
&= -\iint_{S_1} F_3 \, dS + \iint_{S_2} F_3 \, dS - \iint_{S_3} F_1 \, dS \\
&\quad - \iint_{S_4} F_1 \, dS - \iint_{S_5} F_2 \, dS - \iint_{S_6} F_2 \, dS \\
&= -\int_0^1 \int_0^1 2z \, dx \, dy + \int_0^1 \int_0^1 2z \, dx \, dy - \int_0^1 \int_0^1 2x \, dy \, dz \\
&\quad + \int_0^1 \int_0^1 2x \, dy \, dz - \int_0^1 \int_0^1 3y \, dx \, dz + \int_0^1 \int_0^1 3y \, dx \, dz \\
&= 0 + 2\int_0^1 \int_0^1 \, dx \, dy + 0 + 2\int_0^1 \int_0^1 \, dy \, dz + 0 + 3\int_0^1 \int_0^1 \, dx \, dz \\
&= 2 + 2 + 3 \\
&= 7
\end{aligned}$$

where from line 3 to 4, we have plugged in the equation of the plane into the integrand, and we interpret the double integrals as the areas of the unit square $[0,1] \times [0,1]$, which is just 1.

□

2. Use the Divergence theorem to calculate the flux of $\mathbf{F} = (x - 4y, y - 7z, z - 5x)$ out of the unit sphere.

***Solution*:** We want to choose the easiest route to the answer. This will be the case if we choose to compute the volume integral. First compute the divergence of $\mathbf{F}$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x - 4y, y - 7z, z - 5x) \\ &= 1 + 1 + 1 \\ &= 3\end{aligned}$$

Computing the volume integral, we have

$$\begin{aligned}\iiint_W (\nabla \cdot \mathbf{F})\ dV &= \iiint_W 3\ dV \\ &= 3 \iiint_W\ dV \\ &= 3\left(\frac{4}{3}\pi\right) = 4\pi\end{aligned}$$

where we interpret the volume integral $\iiint_W\ dV$ as the volume of the unit sphere, which is $V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi$. □

3. Let $\mathbf{F}(x, y, z) = (4x^3, 4y^3, 4z^3)$. Evaluate the surface integral of $\mathbf{F}$ over the unit sphere.

***Solution*:** Again, this will be easiest if we choose to compute the volume integral. First compute the divergence of $\mathbf{F}$

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) \\ &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (4x^3, 4y^3, 4z^3) \\ &= 12x^2 + 12y^2 + 12z^2 = 12(x^2 + y^2 + z^2)\end{aligned}$$

Computing the volume integral, we have

$$\begin{aligned}\iiint_W (\nabla \cdot \mathbf{F})\ dV &= 12 \iiint_W (x^2 + y^2 + z^2)\ dV \\ &= 12 \int_0^{2\pi} \int_0^{\pi} \int_0^{1} (r^2)(r^2 \sin(\phi)\ dr\ d\phi\ d\theta \\ &= 12 \left(\int_0^{2\pi} d\theta\right) \left(\int_0^{\pi} \sin(\phi)\ d\phi\right) \left(\int_0^1 r^4\ dr\right) \\ &= 12\,(2\pi)\left(-\cos(\phi)|_0^{\pi}\right)\left(\left.\frac{r^5}{5}\right|_0^1\right) \\ &= 12(2\pi)(2)\left(\frac{1}{3}\right) = \frac{48}{5}\pi\end{aligned}$$

□

4. Let $\mathbf{F}(x, y, z) = (2y, 7z, 8xz)$. Evaluate the surface integral of $\mathbf{F}$ over each region W.

$$\begin{aligned}
&(a)\ x^2 + y^2 \leq z \leq 1 \\
&(b)\ x^2 + y^2 \leq z \leq 1 \quad \text{and } x \geq 0 \\
&(c)\ x^2 + y^2 \leq z \leq 1 \quad \text{and } x \leq 0
\end{aligned}$$

***Solution*:** We will compute all of these surface integrals by using the Divergence theorem, and computing the volume integral instead. The region in (a) is the paraboloid that opens up, where $z \in [0, 1]$. We will use cylindrical coordinates to do the integral, so we will take

$$\begin{aligned}
x &= r\cos(\theta) \\
y &= r\sin(\theta) \\
z &= z \\
dx\ dy\ dz &= r\ dr\ d\theta\ dz
\end{aligned}$$

Note that $z \in [0, 1]$ for (a), (b), and (c). Also, we have $z = x^2 + y^2 = r^2$, which means $r \in [0, \sqrt{z}]$ for (a), (b), and (c) as well. The values of θ will change for each question. We also compute the divergence of $\mathbf{F}$, as it is the same for each part:

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (F_1, F_2, F_3) \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (2y, 7z, 8xz) \\
&= 0 + 0 + 8x = 8x = 8r\cos(\theta)
\end{aligned}$$

(a) For the region W, we will take the whole paraboloid bounded between $z = 0$ and $z = 1$. Using Divergence theorem

$$\begin{aligned}
\iiint_W (\nabla \cdot \mathbf{F})\ dV &= 8\iiint_W (r\cos(\theta))r\ dr\ d\theta\ dz \\
&= 8\int_0^{2\pi}\int_0^1\int_0^{\sqrt{z}} r^2\cos(\theta)\ dr\ dz\ d\theta \\
&= 8\int_0^{2\pi}\int_0^1 \frac{z^{\frac{3}{2}}}{3}\cos(\theta)\ dz\ d\theta \\
&= 8\left(\int_0^{2\pi}\cos(\theta)\ d\theta\right)\left(\int_0^1 \frac{z^{\frac{3}{2}}}{3}\ dz\right) \\
&= 8\left(\sin(\theta)|_0^{2\pi}\right)\left(\frac{2}{15}r^{\frac{5}{2}}\Big|_0^1\right) \\
&= 8(0)\left(\frac{2}{15}\right) = 0
\end{aligned}$$

(b) For the region W, we will take the half of the paraboloid bounded between

$\theta = -\frac{\pi}{2}$ and $\theta = \frac{\pi}{2}$. Using Divergence theorem

$$\begin{aligned}
\iiint_W (\nabla \cdot \mathbf{F})\ dV &= 8 \iiint_W (r\cos(\theta))r\ dr\ d\theta\ dz \\
&= 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{z}} r^2 \cos(\theta)\ dr\ dz\ d\theta \\
&= 8 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \frac{z^{\frac{3}{2}}}{3} \cos(\theta)\ dz\ d\theta \\
&= 8 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(\theta)\ d\theta \right) \left(\int_0^1 \frac{z^{\frac{3}{2}}}{3}\ dz \right) \\
&= 8 \left(\sin(\theta)|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right) \left(\frac{2}{15} r^{\frac{5}{2}} \Big|_0^1 \right) \\
&= 8(2) \left(\frac{2}{15} \right) = \frac{32}{15}
\end{aligned}$$

(c) For the region W, we will take the half of the paraboloid bounded between $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Using Divergence theorem

$$\begin{aligned}
\iiint_W (\nabla \cdot \mathbf{F})\ dV &= 8 \iiint_W (r\cos(\theta))r\ dr\ d\theta\ dz \\
&= 8 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^1 \int_0^{\sqrt{z}} r^2 \cos(\theta)\ dr\ dz\ d\theta \\
&= 8 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^1 \frac{z^{\frac{3}{2}}}{3} \cos(\theta)\ dz\ d\theta \\
&= 8 \left(\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(\theta)\ d\theta \right) \left(\int_0^1 \frac{z^{\frac{3}{2}}}{3}\ dz \right) \\
&= 8 \left(\sin(\theta)|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \right) \left(\frac{2}{15} r^{\frac{5}{2}} \Big|_0^1 \right) \\
&= 8(-2) \left(\frac{2}{15} \right) = \frac{32}{15}
\end{aligned}$$

□

5. Find the flux of the vector field $\mathbf{F} = (x - y^3, 6y, x^4)$ out of the rectangular solid $[0,1] \times [1,2] \times [1,4]$.

Solution: Again, this will be easiest if we choose to compute the volume integral. First compute the divergence of $\mathbf{F}$

$$\begin{aligned}
\nabla \cdot \mathbf{F} &= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (F_1, F_2, F_3) \\
&= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot (x - y^3, 6y, x^4) \\
&= 1 + 6 + 0 = 7
\end{aligned}$$

Computing the volume integral, we have

$$\begin{aligned}\iiint_W (\nabla \cdot \mathbf{F})\ dV &= 7 \iiint_W \ dV \\ &= 7(1)(1)(3) \\ &= 21\end{aligned}$$

where we interpret the volume integral $\iiint_W \ dV$ as the volume of the rectangular box, which is the product of length, width, and height. $\square$