MATH 146A

HW#4 Solutions

Section 7.7 #16, 17

Section 7.9 #1, 4, 11

Section 7.7

16) Show that if A is a diagonal matrix with diagonal elements $a_1, a_2, \ldots, a_n$, then $\exp(At)$ is also a diagonal matrix with diagonal elements $\exp(a_1 t), \exp(a_2 t), \ldots, \exp(a_n t)$.

Solution: The definition of matrix exponential is given by eq. 23, p. 424. For an $n \times n$ matrix A,

$$\exp(At) = I + \sum_{j=1}^{\infty} \frac{A^j t^j}{j!} \qquad (23)$$

where I is the $n \times n$ identity matrix. Now, let A be a diagonal matrix given by

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

Using the definition of matrix exponential (23), we have

$$\exp(At) = I + At + \frac{A^2}{2!}t^2 + \frac{A^3}{3!}t^3 + \cdots$$

$$= \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \\ & & & 1 \end{pmatrix} + \begin{pmatrix} a_1 t & & & \\ & a_2 t & & \\ & & \ddots & \\ & & & a_n t \end{pmatrix} + \begin{pmatrix} \frac{a_1^2 t^2}{2!} & & & \\ & \frac{a_2^2 t^2}{2!} & & \\ & & \ddots & \\ & & & \frac{a_n^2 t^2}{2!} \end{pmatrix} + \begin{pmatrix} \frac{a_1^3 t^3}{3!} & & & \\ & \frac{a_2^3 t^3}{3!} & & \\ & & \ddots & \\ & & & \frac{a_n^3 t^3}{3!} \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} 1 + a_1 t + \frac{a_1^2 t^2}{2!} + \frac{a_1^3 t^3}{3!} + \cdots & & & \\ & 1 + a_2 t + \frac{a_2^2 t^2}{2!} + \frac{a_2^3 t^3}{3!} + \cdots & & \\ & & \ddots & \\ & & & 1 + a_n t + \frac{a_n^2 t^2}{2!} + \frac{a_n^3 t^3}{3!} + \cdots \end{pmatrix}$$

$$= \begin{pmatrix} e^{a_1 t} & & & \\ & e^{a_2 t} & & \\ & & \ddots & \\ & & & e^{a_n t} \end{pmatrix}$$

which is what we wanted to show ∎

17) Consider an oscillator satisfying the initial value problem

$$u'' + \omega^2 u = 0, \quad u(0) = u_0, \quad u'(0) = v_0 \quad (i)$$

(a) Let $x_1 = u$, $x_2 = u'$ and transform (i) into the form

$$\vec{x}' = A\vec{x}, \quad \vec{x}(0) = x^0$$

Solution: If $x_1 = u$, $x_2 = u' \Rightarrow x_2' = u''$. Then (i) becomes

$$x_2' + \omega^2 x_1 = 0, \quad x_1(0) = u_0, \quad x_2(0) = v_0$$

$$\Rightarrow x_1' = x_2, \quad x_2' = -\omega^2 x_1$$

$$\Rightarrow \boxed{\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}}$$

$$\text{so } A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}$$

(b) By using eq. (23) show $\exp(At) = I\cos(\omega t) + A\dfrac{\sin(\omega t)}{\omega}$

Solution: Since we need A^2, A^3, A^4, A^5, etc. for eq. (23), compute the first few matrices

$$A^2 = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2 I$$

$$A^3 = \begin{pmatrix} 0 & -\omega^2 \\ \omega^4 & 0 \end{pmatrix} = -\omega^2 A$$

$$A^4 = \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^4 \end{pmatrix} = \omega^4 I$$

note the pattern $\xrightarrow{\text{generalized}}$

$$A^{2k} = (-1)^k \omega^{2k} I$$

$$A^{2k+1} = (-1)^k \omega^{2k} A$$

for $k = 0, 1, 2, 3, \ldots$

Using (23), we have

$$e^{At} = \exp(At) = I + \sum_{n=1}^{\infty} \frac{A^n t^n}{n!}$$

$$= \sum_{k=0}^{\infty} \left[(-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} I + (-1)^k \frac{\omega^{2k} t^{2k+1}}{(2k+1)!} A \right]$$

(cont.)

$$= \left(\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k} t^{2k}}{(2k)!} \right) I + \frac{1}{\omega} \left(\sum_{k=0}^{\infty} (-1)^k \frac{\omega^{2k+1} t^{2k+1}}{(2k+1)!} \right) A$$

$$= I \cos(\omega t) + A \frac{\sin(\omega t)}{\omega} \quad \blacksquare$$

(c) Find the solution of the initial value problem from part (a).

<u>Solution:</u> By eq. (28), p. 424, the solution is given by

$$\vec{x} = \exp(At)\vec{x}^0$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left[I \cos(\omega t) + A \frac{\sin(\omega t)}{\omega} \right] \cdot \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \cos(\omega t) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \frac{1}{\omega} \sin(\omega t) \begin{pmatrix} v_0 \\ -\omega^2 u_0 \end{pmatrix}}$$

Section 7.9

1) Find the general solution to $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$

<u>Solution:</u> First solve the homogeneous system $\vec{x}' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}$

$$\Rightarrow A = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \Rightarrow \lambda_1 = 1, \lambda_2 = -1 \text{ eigenvalues}$$

$$\vec{\xi}_1 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{\xi}_2 = c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ eigenvectors}$$

$$\Rightarrow \Psi(t) = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \Rightarrow \Psi^{-1}(t) = \frac{1}{2} \begin{pmatrix} 3e^{-t} & -e^{-t} \\ -e^t & e^t \end{pmatrix} ; \vec{g}(t) = \begin{pmatrix} e^t \\ t \end{pmatrix}$$

Recall from section 7.7, that $\Psi(t) \cdot \vec{c}$ is the solution to the homogeneous system $\vec{x}' = A\vec{x}$. The general solution is given by eq. (29), p. 444

$$\vec{x} = \Psi \cdot \vec{c} + \Psi(t) \int \Psi^{-1}(s) \vec{g}(s)\, ds$$

Thus we only need to compute the particular solution piece: $\Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds$

$$\Rightarrow \Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds = \begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \int^t \frac{1}{2}\begin{pmatrix} 3e^{-s} & -e^{-s} \\ -e^{s} & e^{s} \end{pmatrix}\begin{pmatrix} e^{s} \\ s \end{pmatrix} ds$$

$$= \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix} \int^t \begin{pmatrix} 3-se^{-s} \\ -e^{2s}+se^{s} \end{pmatrix} ds$$

$$= \frac{1}{2}\begin{pmatrix} e^t & e^{-t} \\ e^t & 3e^{-t} \end{pmatrix}\begin{pmatrix} 3t + e^{-t}(t+1) \\ -\frac{1}{2}e^{2t} + e^{t}(t-1) \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 3te^t + (t+1) - \frac{1}{2}e^t + (t-1) \\ 3te^t + (t+1) - \frac{3}{2}e^t + 3(t-1) \end{pmatrix}$$

$$\Rightarrow \Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds = \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}te^t - \frac{1}{4}\begin{pmatrix}1\\3\end{pmatrix}e^t + \begin{pmatrix}1\\2\end{pmatrix}t - \begin{pmatrix}0\\1\end{pmatrix}$$

$$\Rightarrow \boxed{\vec{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t} + \frac{3}{2}\begin{pmatrix}1\\1\end{pmatrix}te^t - \frac{1}{4}\begin{pmatrix}1\\3\end{pmatrix}e^t + \begin{pmatrix}1\\2\end{pmatrix}t - \begin{pmatrix}0\\1\end{pmatrix}}$$

4) Find the general solution of $\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$

<u>Solution</u>: Let $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix} \Rightarrow \lambda_1 = -3,\ \lambda_2 = 2$ eigenvalues

$\vec{\xi}_1 = c_1\begin{pmatrix}1\\-4\end{pmatrix},\ \vec{\xi}_2 = c_2\begin{pmatrix}1\\1\end{pmatrix}$ eigenvectors

$$\Rightarrow \Psi(t) = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t} \end{pmatrix} \Rightarrow \Psi^{-1}(t) = \frac{1}{5}\begin{pmatrix} e^{3t} & -e^{3t} \\ 4e^{-2t} & e^{-2t} \end{pmatrix}$$

$$\vec{g}(t) = \begin{pmatrix} e^{-2t} \\ -2e^t \end{pmatrix}$$

Now compute the particular solution

$$\Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds$$

$\Rightarrow$

$$\Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds = \begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t}\end{pmatrix}\int^t \frac{1}{5}\begin{pmatrix} e^{3s} & -e^{3s} \\ 4e^{-2s} & e^{-2s}\end{pmatrix}\begin{pmatrix} e^{-2s} \\ -2e^{s}\end{pmatrix} ds$$

$$= \frac{1}{5}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t}\end{pmatrix}\int^t \begin{pmatrix} e^{s} + 2e^{4s} \\ 4e^{-4s} - 2e^{-s}\end{pmatrix} ds$$

$$= \frac{1}{5}\begin{pmatrix} e^{-3t} & e^{2t} \\ -4e^{-3t} & e^{2t}\end{pmatrix}\begin{pmatrix} e^{t} + \frac{1}{2}e^{4t} \\ -e^{-4t} + 2e^{-t}\end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} e^{-2t} + \frac{1}{2}e^{t} - e^{-2t} + 2e^{t} \\ -4e^{-2t} - 2e^{t} - e^{-2t} + 2e^{t}\end{pmatrix} = \frac{1}{5}\begin{pmatrix} \frac{5}{2}e^{t} \\ -5e^{-2t}\end{pmatrix}$$

$$\Rightarrow \Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds = -\begin{pmatrix} 0 \\ 1\end{pmatrix}e^{-2t} + \frac{1}{2}\begin{pmatrix} 1 \\ 0\end{pmatrix}e^{t}$$

$$\Rightarrow \boxed{\vec{x} = c_1\begin{pmatrix} 1 \\ -4\end{pmatrix}e^{-3t} + c_2\begin{pmatrix} 1 \\ 1\end{pmatrix}e^{2t} - \begin{pmatrix} 0 \\ 1\end{pmatrix}e^{-2t} + \frac{1}{2}\begin{pmatrix} 1 \\ 0\end{pmatrix}e^{t}}$$

11) Find the general solution of $\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2\end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ \cos(t)\end{pmatrix}$ $0 < t < \pi$

<u>Solution</u>: $A = \begin{pmatrix} 2 & -5 \\ 1 & -2\end{pmatrix} \Rightarrow \lambda = \pm i$ complex eigenvalues

Problem #3 from 7.6 gives the general solution to the homogeneous problem, hence we have the fundamental matrix

$$\Psi(t) = \begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t) + \sin(t) & -\cos(t) + 2\sin(t)\end{pmatrix}, \quad \vec{g}(t) = \begin{pmatrix} 0 \\ \cos(t)\end{pmatrix}$$

$$\Rightarrow \Psi^{-1}(t) = -\frac{1}{5}\begin{pmatrix} -\cos(t) + 2\sin(t) & -5\sin(t) \\ -2\cos(t) - \sin(t) & 5\cos(t)\end{pmatrix} = \frac{1}{5}\begin{pmatrix} \cos(t) - 2\sin(t) & 5\sin(t) \\ 2\cos(t) + \sin(t) & -5\cos(t)\end{pmatrix}$$

$$\Psi(t)\int^t \Psi^{-1}(s)\vec{g}(s)\,ds =$$

$$\begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t)+\sin(t) & -\cos(t)+2\sin(t) \end{pmatrix} \int^t \frac{1}{5}\begin{pmatrix} \cos(s)-2\sin(s) & 5\sin(s) \\ 2\cos(s)+\sin(s) & -5\cos(s) \end{pmatrix}\begin{pmatrix} 0 \\ \cos(s) \end{pmatrix} ds$$

$$= \begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t)+\sin(t) & -\cos(t)+2\sin(t) \end{pmatrix} \int^t \begin{pmatrix} \cos(s)\sin(s) \\ -\cos^2(s) \end{pmatrix} ds$$

or $-\frac{1}{2}\cos^2(t)$ (*)

$$= \begin{pmatrix} 5\cos(t) & 5\sin(t) \\ 2\cos(t)+\sin(t) & -\cos(t)+2\sin(t) \end{pmatrix} \begin{pmatrix} \frac{1}{2}\sin^2(t) \\ -\frac{1}{2}\cos(t)\sin(t) - \frac{1}{2}t \end{pmatrix}$$

$$= \begin{pmatrix} \cancel{\frac{5}{2}\cos(t)\sin^2(t)} - \cancel{\frac{5}{2}\cos(t)\sin^2(t)} - \frac{5}{2}t\sin(t) \\ \cancel{\cos(t)\sin^2(t)} + \underbrace{\frac{1}{2}\sin^3(t) + \frac{1}{2}\cos^2(t)\sin(t)} + \frac{1}{2}t\cos(t) - \cancel{\cos(t)\sin^2(t)} - t\sin(t) \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{5}{2}t\sin(t) \\ \frac{1}{2}\sin(t) + \frac{1}{2}t\cos(t) - t\sin(t) \end{pmatrix} \quad \underset{\text{using } (*)}{\text{or}} \quad \begin{pmatrix} -\frac{5}{2}t\sin(t) - \frac{5}{2}\cos(t) \\ -t\sin(t) + \frac{1}{2}\cos(t) - \cos(t) \end{pmatrix}$$

Solutions could be

$$\vec{X} = c_1\begin{pmatrix} 5\cos(t) \\ 2\cos(t)+\sin(t) \end{pmatrix} + c_2\begin{pmatrix} 5\sin(t) \\ -\cos(t)+2\sin(t) \end{pmatrix} - t\sin(t)\begin{pmatrix} 5/2 \\ 1 \end{pmatrix} + t\cos(t)\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} + \sin(t)\begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

or

$$\vec{X} = c_1\begin{pmatrix} 5\cos(t) \\ 2\cos(t)+\sin(t) \end{pmatrix} + c_2\begin{pmatrix} 5\sin(t) \\ -\cos(t)+2\sin(t) \end{pmatrix} - t\sin(t)\begin{pmatrix} 5/2 \\ 1 \end{pmatrix} + t\cos(t)\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} - \cos(t)\begin{pmatrix} 5/2 \\ 1 \end{pmatrix}$$