MATH 046 020-QUIZ 8, SPRING 2018

Name: KEY

1 (5 pts). We know that $\ddot{x} - 6\dot{x} + 9x = 0$ has two linearly independent solutions as e^{3t} and te^{3t}. Find a particular solution of the nonhomogeneous equation

$$\ddot{x} - 6\dot{x} + 9x = \frac{e^{3t}}{t^2}$$

From directions, homogeneous solution is

$$x_h(t) = c_1 e^{3t} + c_2 t e^{3t}$$

$\Rightarrow$ We take particular solution as

$$x_p(t) = V_1(t) e^{3t} + V_2(t) t e^{3t}$$

$\Rightarrow$ By variation of parameters

$$① \quad V_1'(t)(e^{3t}) + V_2'(t)(te^{3t}) = 0$$

$$② \quad V_1'(t)(3e^{3t}) + V_2'(t)(e^{3t} + 3te^{3t}) = \frac{e^{3t}}{t^2}$$

$\Rightarrow$ Multiply -3 to ① and then do $-3 \cdot ① + ②$

$$\Rightarrow \quad V_2'(t) e^{3t} = \frac{e^{3t}}{t^2} \Rightarrow V_2'(t) = \frac{1}{t^2} \text{ and } V_1'(t) = -\frac{1}{t}$$

$$\Rightarrow V_1(t) = \int V_1'(t)\,dt = \int -\frac{1}{t}\,dt = -\ln|t|$$

$$V_2(t) = \int V_2'(t)\,dt = \int \frac{1}{t^2}\,dt = -\frac{1}{t}$$

$$\Rightarrow \boxed{x(t) = (c_1 - 1)e^{3t} + c_2 t e^{3t} - e^{3t}\ln|t|}$$

2 (5 pts). Find the general solution of the equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \ln x,$$

provided that two solutions to the associated homogeneous equation are known to be 1 and x^2.

$\Rightarrow Y_h(x) = C_1 X + C_2 \frac{1}{X}$ and $Y_p(x) = V_1(x) X + V_2(x) \frac{1}{X}$

$$\Rightarrow \begin{cases} V_1'(x) X + V_2'(x) \frac{1}{X} = 0 \\ V_1'(x)(1) + V_2'(x)\left(-\frac{1}{X^2}\right) = \ln(x) \end{cases}$$

Multiply bottom equation by $-X$ and add to get

$$V_2'(x)\left(\frac{2}{X}\right) = -X \ln(x)$$

$$V_2'(x) = -\frac{1}{2}X^2 \ln(x) \quad \text{and } V_1'(x) = \frac{1}{2}\ln(x)$$

$$\Rightarrow V_1(x) = \int V_1'(x)\,dx = \int \frac{1}{2}\ln(x)\,dx = -\frac{1}{2}X + \frac{1}{2}X\ln|X|$$

$$V_2(x) = \int V_2'(x)\,dx = \int -\frac{1}{2}x^2\ln(x)\,dx = \frac{1}{18}x^3 - \frac{1}{6}x^3\ln|x|$$

$$\Rightarrow \boxed{Y(x) = C_1 X + C_2\frac{1}{X} + \frac{1}{3}x^2\ln|x| - \frac{4}{9}x^2}$$