MATH 046 - Spring 2018

Worked Problems - Chapter 1

1. Which of the functions below are solutions to the differential equation

$$y'' - 4y' + 4y = e^x$$
$$(a)\ y(x) = e^x \quad (b)\ y(x) = e^{2x} \quad (c)\ y(x) = e^x + e^{2x} \quad (d)\ xe^{2x} + e^x$$

***Solution*:** All we have to do to check if a function is a solution to the differential equation is to plug it into the differential equation, and check that the left and right sides are equal.

For (a),

$$\begin{aligned} y'' - 4y' + 4y &= e^x \\ e^x - 4e^x + 4e^x &\overset{?}{=} e^x. \\ e^x &= e^x \end{aligned}$$

So $y(x) = e^x$ is a solution.

For (b),

$$\begin{aligned} y'' - 4y' + 4y &= e^x \\ 4e^{2x} - 8e^{2x} + 4e^{2x} &\overset{?}{=} e^x. \\ 0 &= e^x. \end{aligned}$$

So $y(x) = e^{2x}$ is not a solution.

For (c),

$$\begin{aligned} y'' - 4y' + 4y &= e^x \\ \left(e^x + 4e^{2x}\right) - 4\left(e^x + 2e^{2x}\right) + 4\left(e^x + e^{2x}\right) &\overset{?}{=} e^x \\ e^x + 4e^{2x} - 4e^x - 8e^{2x} + 4e^x + 4e^{2x} &\overset{?}{=} e^x \\ e^x + 4e^{2x} - 8e^{2x} + 4e^{2x} &\overset{?}{=} e^x \\ e^x &= e^x. \end{aligned}$$

So $y(x) = e^x + e^{2x}$ is a solution.

For (d),

$$\begin{aligned} y'' - 4y' + 4y &= e^x \\ \left(4e^{2x} + 4xe^{2x} + e^x\right) - 4\left(e^{2x} + 2xe^{2x} + e^x\right) + 4\left(xe^{2x} + e^x\right) &\overset{?}{=} e^x \\ 4e^{2x} + 4xe^{2x} + e^x - 4e^{2x} - 8xe^{2x} - 4e^x + 4xe^{2x} + 4e^x &\overset{?}{=} e^x \\ 4e^{2x} + e^x - 4e^{2x} - 4e^x + 4e^x &\overset{?}{=} e^x \\ e^x &= e^x. \end{aligned}$$

So $y(x) = e^x + e^{2x}$ is a solution. □

2. Find the value of c so that $y(x) = c(1-x^2)$ satisfies the given initial condition.

$$y(2) = 1$$

***Solution*:** For these types of questions, we evaluate $y(x)$ at the point they give us, and set the function equal to the value given. Then solve for c.

$$\begin{aligned} y(x) &= c(1-x^2) \\ y(2) &= c(1-2^2) \\ 1 &= -3c \\ c &= -\frac{1}{3}. \end{aligned}$$

□

3. Find c_1 and c_2 so that $y(x) = c_1 \sin(x) + c_2 \cos(x)$ will satisfy the conditions

$$y(0) = 1 \qquad y'(\pi) = 1$$

Determine whether the given conditions are initial conditions or boundary conditions.

***Solution*:** We can proceed in the same way as the previous problem, except now there are two constants. So using the first condition

$$\begin{aligned} y(x) &= c_1 \sin(x) + c_2 \cos(x) \\ y(0) &= c_1 \sin(0) + c_2 \cos(0) \\ 1 &= c_2. \end{aligned}$$

So thus far, we must have that $y(x) = c_1 \sin(x) + \cos(x)$ in order to satisfy the first condition. To use the second condition we need the derivative of $y(x)$, which is $y'(x) = c_1 \cos(x) - \sin(x)$. Now we proceed with the second condition

$$\begin{aligned} y'(x) &= c_1 \cos(x) - \sin(x) \\ y'(\pi) &= c_1 \cos(\pi) - \sin(\pi) \\ 1 &= -c_1 \\ c_1 &= -1. \end{aligned}$$

Therefore we have that $c_1 = -1$ and $c_2 = 1$ satisfy the conditions. These are boundary conditions, since we are evaluating the functions $y(x)$ and $y'(x)$ at different points. □

4. Find c_1 and c_2 so that $y(x) = c_1 \sin(x) + c_2 \cos(x)$ will satisfy the conditions

$$y(0) = 0 \qquad y'\left(\frac{\pi}{2}\right) = 1$$

Determine whether the given conditions are initial conditions or boundary conditions.

***Solution*:** We can proceed in the same way as the previous problem, except now there are two constants. So using the first condition

$$\begin{aligned} y(x) &= c_1 \sin(x) + c_2 \cos(x) \\ y(0) &= c_1 \sin(0) + c_2 \cos(0) \\ 0 &= c_2. \end{aligned}$$

So thus far, we must have that $y(x) = c_1 \sin(x)$ in order to satisfy the first condition. To use the second condition we need the derivative of $y(x)$, which is $y'(x) = c_1 \cos(x)$. Now we proceed with the second condition

$$\begin{aligned} y'(x) &= c_1 \cos(x) \\ y'\left(\frac{\pi}{2}\right) &= c_1 \cos\left(\frac{\pi}{2}\right) \\ 1 &= 0. \end{aligned}$$

The value of 1 cannot be equal to 0, so the conditions are satisfied for no values of c_1 and c_2. These are boundary conditions, since we are evaluating the functions $y(x)$ and $y'(x)$ at different points. $\square$