MATH 046 - Spring 2018

Worked Problems - Chapter 10

1. Solve the following differential equation

$$y''' - 2y'' - y' + 2y = 0$$

***Solution*:** Note that the ODE is homogeneous with constant coefficients, so we can use the characteristic equation. Note that when we plug in $\lambda = 1$, we get $0 = 0$, so $\lambda = 1$ is a root. Using polynomial long division or synthetic division, we can break down the polynomial. Therefore, we get

$$\begin{aligned} y''' - 2y'' - y' + 2y &= 0 \\ \lambda^3 - 2\lambda^2 - \lambda + 2 &= 0 \\ (\lambda - 1)(\lambda^2 - \lambda - 2) &= 0 \\ (\lambda - 1)(\lambda + 1)(\lambda - 2) &= 0 \\ \lambda &= 1, 2, -1 \end{aligned}$$

So we have three distinct real roots, so then the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 e^{2x}$$

□

2. Solve the following differential equation

$$y^{(4)} + 2y'' + y = 0$$

***Solution*:** Note that the ODE is homogeneous with constant coefficients, so we can use the characteristic equation. Here we do a substitution trick to reduce the 4th order polynomial to a quadratic. Therefore, we get

$$\begin{aligned} y^{(4)} + 2y'' + y &= 0 \\ \lambda^4 + 2\lambda^2 + 1 &= 0 \\ \mu^2 + 2\mu + 1 &= 0 \qquad \text{use } \mu = \lambda^2 \\ (\mu + 1)(\mu + 1) &= 0 \\ (\lambda^2 + 1)(\lambda^2 + 1) &= 0 \\ \lambda &= \pm i, \pm i \end{aligned}$$

So we have repeated complex roots, so then the general solution is

$$y(x) = (c_1 + c_2 x)\cos(x) + (c_3 + c_4 x)\sin(x)$$

where we have used the repeated roots factor $(c_1 + c_2 x)$, as we did for the case with real repeated roots. □

3. Solve the following differential equation

$$y^{(4)} - y = 0$$

***Solution*:** Note that the ODE is homogeneous with constant coefficients, so we can use the characteristic equation. Note that when we plug in $\lambda = 1$, we get $0 = 0$, so $\lambda = 1$ is a root. Also note that $\lambda = -1$ also works, so it is also a root. Using polynomial long division or synthetic division, we can break down the polynomial. Therefore, we get

$$\begin{aligned} y^{(4)} - y &= 0 \\ \lambda^4 - 1 &= 0 \\ (\lambda - 1)(\lambda^3 + \lambda^2 + \lambda + 1) &= 0 \\ (\lambda - 1)(\lambda + 1)(\lambda^2 + 1) &= 0 \\ \lambda &= 1, -1, \pm i \end{aligned}$$

So we have two distinct real roots, and a pair of complex roots, so then the general solution is

$$y(x) = c_1 e^{-x} + c_2 e^{x} + c_3 \cos(x) + c_4 \sin(x)$$

□

4. Solve the following differential equation

$$y^{(4)} + 5y''' = 0$$

***Solution*:** Note that the ODE is homogeneous with constant coefficients, so we can use the characteristic equation. Note that when we plug in $\lambda = 1$, we get $0 = 0$, so $\lambda = 1$ is a root. Also note that $\lambda = -1$ also works, so it is also a root. Using polynomial long division or synthetic division, we can break down the polynomial. Therefore, we get

$$\begin{aligned} y^{(4)} + 5y''' &= 0 \\ \lambda^4 + 5\lambda^3 &= 0 \\ \lambda^3(\lambda + 5) &= 0 \\ \lambda &= 0, 0, 0, -5 \end{aligned}$$

So we have a real repeated root that appears 3 times, and another real root, so then the general solution is

$$y(x) = c_1 + c_2 x + c_3 x^2 + c_4 e^{-5x}$$

□

5. Consider a pendulum with length L (meters) and angle θ (radians) from the vertical to the pendulum. By Newtons second law for rotation, we showed in class that θ as a function of time satisfies the differential equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0 \tag{1}$$

where $g = 10\text{m/sec}^2$ is the acceleration due to gravity. For small values of θ we can use the approximation $\sin(\theta) \approx \theta$ and with that substitution, the differential equation becomes a linear equation:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \tag{2}$$

The linear equation (2) is called the linearized equation of the nonlinear equation (1). Suppose the length is 0.4 meters, the initial angle 0.1 radians, and the initial angular velocity $\frac{d\theta}{dt} = 0.5$ radians/sec. Use the linearized equation (2) to answer the questions below.

(a) Find the function $\theta(t)$ which describes the motion of the pendulum with the given initial conditions.

(b) What is the maximum angle (in radians) from vertical?

(c) What is the period of the pendulum, that is the time for one swing back and forth?

(d) How long after reaching its maximum angle until the pendulum reaches maximum deflection in the other direction? (Hint: think about its relation with the period.)

***Solution*:** (a) Note that the ODE is homogeneous with constant coefficients, so we can use the characteristic equation. Therefore, we get

$$\begin{aligned}
\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta &= 0 \\
\lambda^2 + \frac{g}{L} &= 0 \\
\lambda &= \pm\sqrt{-\frac{g}{L}} \\
&= \pm\sqrt{-\frac{10}{0.4}} \\
&= \pm 5i
\end{aligned}$$

So we have a pair of complex roots, so then the general solution is

$$\theta(t) = c_1\cos(5t) + c_2\sin(5t)$$

Using the initial conditions, we have $\theta(0) = \frac{1}{10}$ and $\frac{d\theta}{dt}(0) = \frac{1}{2}$, which means that we have

$$\begin{aligned}
\theta(0) &= c_1\cos(0) + c_2\sin(0) \\
\frac{1}{10} &= c_1 \\
&\text{and} \\
\frac{d\theta}{dt}(0) &= -\frac{1}{2}\sin(0) + 5c_2\cos(0) \\
\frac{1}{2} &= 5c_2 \\
\frac{1}{10} &= c_2
\end{aligned}$$

So then the solution to the initial value problem is

$$\theta(t) = \frac{1}{10}\cos(5t) + \frac{1}{10}\sin(5t)$$

(b) The maximum angle (in radians) from vertical is the maximum of the function $\theta(t)$. So note that

$$\begin{aligned}
\theta(t) &= \frac{1}{10}\cos(5t) + \frac{1}{10}\sin(5t) \\
\theta'(t) &= -\frac{1}{2}\sin(5t) + \frac{1}{2}\cos(5t) \\
0 &= -\frac{1}{2}\sin(5t) + \frac{1}{2}\cos(5t) \\
\tan(5t) &= 1 \\
t &= \frac{\pi}{20}(4n+1)
\end{aligned}$$

where we have found the t values of the critical points. To find the value, we just evaluate the function $\theta(t)$, so that we see a maximum is $\theta\left(\frac{\pi}{20}\right) = \frac{1}{5\sqrt{2}} = \frac{\sqrt{2}}{10}$ radians.

(c) Both trigonometric functions have the same frequency, $\omega = 5$. The sum does not change the period, so we use the period formula

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{5} = \frac{2}{5}\pi$$

So we get $\frac{2}{5}\pi$ seconds.

(d) The hint says to think about its relation with the period. We know that the period is the amount of time it takes for the pendulum to return to its initial position, so that is $\frac{2}{5}\pi$. The question is essentially asking how much time passes between the maximum and minimum of function $\theta(t)$. We already know the maximum occurs at $t = \frac{\pi}{20}$. Now the time between the maximum and minimum over one period of a sine/cosine function is one half the period, or $\frac{\pi}{5}$, but we can show this. So, the t_{max} which is the time where the $\theta(t)$ function is maximum is at $t_{\text{max}} = \frac{\pi}{20}$, then we get that the minimum occurs at

$$t_{\text{min}} = \frac{\pi}{20} + \frac{1}{2}\left(\frac{2}{5}\pi\right) = \frac{\pi}{20} + \frac{1}{5}\pi = \frac{5}{20}\pi = \frac{\pi}{4}$$

This is just the result of taking $n = 1$ in the solution to part (b), which gives the t values of the critical points. You can verify the answer by taking the difference between the t_{min} and t_{max}, so

$$\Delta t = t_{\text{min}} - t_{\text{max}} = \frac{\pi}{4} - \frac{\pi}{20} = \frac{\pi}{5}$$

□