MATH 046 - Spring 2018

Worked Problems - Chapter 12

1. Use variation of parameters to find the general solutions of the following differential equation

$$y'' - 2y' + y = \frac{e^x}{x^5}$$

***Solution*:** Note that homogeneous solution to the ODE is $y_h(x) = (c_1 + c_2 x)e^x$. Therefore, the particular solution is chosen as

$$y_p(x) = v_1(x)e^x + v_2(x)xe^x$$

and the system of equations from the method of variation of parameters gives

$$\begin{aligned} v_1'(x)\,(e^x) + v_2'(x)\,(xe^x) &= 0 \\ v_1'(x)\,(e^x) + v_2'(x)\,(e^x + xe^x) &= \frac{e^x}{x^5} \end{aligned}$$

where we used $\phi(x) = \dfrac{e^x}{x^5}$ to be the non-homogeneous factor in the ODE. Subtracting the top equation from the bottom equation in the above system, we get

$$\begin{aligned} v_2'(x)\,(e^x + xe^x) - v_2'(x)\,(xe^x) &= \frac{e^x}{x^5} \\ v_2'(x)\,(e^x) &= \frac{e^x}{x^5} \\ v_2'(x) &= \frac{1}{x^5} \end{aligned}$$

which means that $v_1'(x) = -\dfrac{1}{x^4}$. Then we solve for $v_1(x)$ and $v_2(x)$ to get

$$\begin{aligned} v_1(x) &= \int v_1'(x)\; dx = \int -\frac{1}{x^4}\; dx = \frac{1}{3x^3} \\ v_2(x) &= \int v_2'(x)\; dx = \int \frac{1}{x^5}\; dx = -\frac{1}{4x^4} \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= (c_1 + c_2 x)e^x + \frac{e^x}{3x^3} - \frac{xe^x}{4x^4} \\ &= (c_1 + c_2 x)e^x + \frac{e^x}{12x^3} \end{aligned}$$

□

2. Use variation of parameters to find the general solutions of the following differential equation

$$y'' + y = \sec(x)$$

***Solution*:** Note that homogeneous solution to the ODE is $y_h(x) = c_1 \cos(x) + c_2 \sin(x)$. Therefore, the particular solution is chosen as

$$y_p(x) = v_1(x)\cos(x) + v_2(x)\sin(x)$$

and the system of equations from the method of variation of parameters gives

$$\begin{aligned} v_1'(x)\,(\cos(x)) + v_2'(x)\,(\sin(x)) &= 0 \\ v_1'(x)\,(-\sin(x)) + v_2'(x)\,(\cos(x)) &= \sec(x) \end{aligned}$$

where we used $\phi(x) = \sec(x)$ to be the non-homogeneous factor in the ODE. Multiplying the top equation by $\sin(x)$ and the bottom equation by $\cos(x)$, then adding the equations together, we get

$$\begin{aligned} v_2'(x)\left(\sin^2(x)\right) + v_2'(x)\left(\cos^2(x)\right) &= \cos(x)\sec(x) \\ v_2'(x)\left(\sin^2(x) + \cos^2(x)\right) &= 1 \\ v_2'(x) &= 1 \end{aligned}$$

which means that $v_1'(x) = -\dfrac{\sin(x)}{\cos(x)} = -\tan(x)$. Then we solve for $v_1(x)$ and $v_2(x)$ to get

$$\begin{aligned} v_1(x) &= \int v_1'(x)\; dx = \int -\tan(x)\; dx = -(-\ln|\cos(x)|) = \ln|\cos(x)| \\ v_2(x) &= \int v_2'(x)\; dx = \int 1\; dx = x \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1\cos(x) + c_2\sin(x) + \cos(x)\ln|\cos(x)| + x\sin(x) \end{aligned}$$

□

3. Use variation of parameters to find the general solutions of the following differential equation

$$y'' + \frac{1}{x}y' - \frac{1}{x^2}y = \ln(x)$$

***Solution*:** From the hint, the homogeneous solution to the ODE is $y_h(x) = c_1 x + c_2 \frac{1}{x}$. Therefore, the particular solution is chosen as

$$y_p(x) = v_1(x)x + v_2(x)\frac{1}{x}$$

and the system of equations from the method of variation of parameters gives

$$\begin{aligned} v_1'(x)\,(x) + v_2'(x)\left(\frac{1}{x}\right) &= 0 \\ v_1'(x)\,(1) + v_2'(x)\left(-\frac{1}{x^2}\right) &= \ln(x) \end{aligned}$$

where we used $\phi(x) = \ln(x)$ to be the non-homogeneous factor in the ODE. Multiplying the bottom equation by $-x$, and then adding the equations together, we get

$$\begin{aligned} v_2'(x)\left(\frac{1}{x}\right) + v_2'(x)\left(\frac{1}{x}\right) &= -x\ln(x) \\ v_2'(x)\left(\frac{2}{x}\right) &= -x\ln(x) \\ v_2'(x) &= -\frac{1}{2}x^2\ln(x) \end{aligned}$$

which means that $v_1'(x) = \dfrac{1}{2}\ln(x)$. Then we solve for $v_1(x)$ and $v_2(x)$ to get

$$\begin{aligned} v_1(x) &= \int v_1'(x)\;dx = \int \frac{1}{2}\ln(x)\;dx = -\frac{1}{2}x + \frac{1}{2}x\ln|x| \\ v_2(x) &= \int v_2'(x)\;dx = \int -\frac{1}{2}x^2\ln(x)\;dx = \frac{1}{18}x^3 - \frac{1}{6}x^3\ln|x| \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 x + c_2\frac{1}{x} + x\left(-\frac{1}{2}x + \frac{1}{2}x\ln|x|\right) + \frac{1}{x}\left(\frac{1}{18}x^3 - \frac{1}{6}x^3\ln|x|\right) \\ &= c_1 x + c_2\frac{1}{x} + \frac{1}{3}x^2\ln|x| - \frac{4}{9}x^2 \end{aligned}$$

□

4. Use variation of parameters to find the general solutions of the following differential equation

$$x^2y'' - xy' = x^3e^x$$

***Solution*:** First, we need to have the ODE in standard form to solve this question. So, we use the ODE form

$$y'' - \frac{1}{x}y' = xe^x$$

From the hint, the homogeneous solution to the ODE is $y_h(x) = c_1 + c_2x^2$. Therefore, the particular solution is chosen as

$$y_p(x) = v_1(x) + v_2(x)x^2$$

and the system of equations from the method of variation of parameters gives

$$\begin{aligned} v_1'(x)\,(1) + v_2'(x)\left(x^2\right) &= 0 \\ 0 + v_2'(x)\,(2x) &= xe^x \end{aligned}$$

where we used $\phi(x) = xe^x$ to be the non-homogeneous factor in the ODE. Solving the bottom equation directly, we get that $v_2'(x) = \frac{1}{2}e^x$. Solving the top equation, we get

$$\begin{aligned} v_1'(x)\,(1) + \left(\frac{1}{2}e^x\right)\left(x^2\right) &= 0 \\ v_1'(x) &= -\frac{1}{2}x^2e^x \end{aligned}$$

Then we solve for $v_1(x)$ and $v_2(x)$ to get

$$\begin{aligned} v_1(x) &= \int v_1'(x)\ dx = \int -\frac{1}{2}x^2e^x\ dx = -\frac{1}{2}e^x(x^2 - 2x + 2) \\ v_2(x) &= \int v_2'(x)\ dx = \int \frac{1}{2}e^x\ dx = \frac{1}{2}e^x \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1 + c_2x^2 + (1)\left(-\frac{1}{2}e^x(x^2 - 2x + 2))\right) + x^2\left(\frac{1}{2}e^x\right) \\ &= c_1 + c_2x^2 + xe^x - e^x \end{aligned}$$

□

5. Use variation of parameters to find the general solutions of the following differential equation

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$$

***Solution*:** Note that homogeneous solution to the ODE is $y_h(x) = (c_1 + c_2 x)e^{3x}$. Therefore, the particular solution is chosen as

$$y_p(x) = v_1(x)e^{3x} + v_2(x)xe^{3x}$$

and the system of equations from the method of variation of parameters gives

$$\begin{aligned} v_1'(x)\left(e^{3x}\right) + v_2'(x)\left(xe^{3x}\right) &= 0 \\ v_1'(x)\left(3e^{3x}\right) + v_2'(x)\left(e^{3x} + 3xe^{3x}\right) &= \frac{e^{3x}}{x^2} \end{aligned}$$

where we used $\phi(x) = \frac{e^{3x}}{x^2}$ to be the non-homogeneous factor in the ODE. Multiplying the top equation by -3, then adding the equations together, we get

$$\begin{aligned} v_2'(x)\left(e^{3x} + 3xe^{3x}\right) + v_2'(x)\left(-3xe^{3x}\right) &= \frac{e^{3x}}{x^2} \\ v_2'(x)\left(e^{3x}\right) &= \frac{e^{3x}}{x^2} \\ v_2'(x) &= \frac{1}{x^2} \end{aligned}$$

which means that $v_1'(x) = -\frac{1}{x}$. Then we solve for $v_1(x)$ and $v_2(x)$ to get

$$\begin{aligned} v_1(x) &= \int v_1'(x)\, dx = \int -\frac{1}{x}\, dx = -\ln|x| \\ v_2(x) &= \int v_2'(x)\, dx = \int \frac{1}{x^2}\, dx = -\frac{1}{x} \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= (c_1 + c_2 x)e^{3x} + e^{3x}(-\ln|x|) + xe^{3x}\left(-\frac{1}{x}\right) \\ &= (c_1 - 1)e^{3x} + c_2 xe^{3x} - e^{3x}\ln|x| \end{aligned}$$

□

6. Use variation of parameters to find the general solutions of the following differential equation

$$y'' - 4y' + 3y = \frac{e^x}{1+e^x}$$

***Solution*:** Note that homogeneous solution to the ODE is $y_h(x) = c_1 e^x + c_2 e^{3x}$. Therefore, the particular solution is chosen as

$$y_p(x) = v_1(x)e^x + v_2(x)e^{3x}$$

and the system of equations from the method of variation of parameters gives

$$\begin{aligned} v_1'(x)\left(e^x\right) + v_2'(x)\left(e^{3x}\right) &= 0 \\ v_1'(x)\left(e^x\right) + v_2'(x)\left(3e^{3x}\right) &= \frac{e^x}{1+e^x} \end{aligned}$$

where we used $\phi(x) = \dfrac{e^x}{1+e^x}$ to be the non-homogeneous factor in the ODE. Multiplying the top equation by -1, then adding the equations together, we get

$$\begin{aligned} v_2'(x)\left(2e^{3x}\right) &= \frac{e^x}{1+e^x} \\ v_2'(x) &= \frac{1}{2}\frac{e^{-2x}}{1+e^x} \end{aligned}$$

which means that $v_1'(x) = -\dfrac{1}{2}\dfrac{1}{1+e^x}$. Then we solve for $v_1(x)$ and $v_2(x)$ to get

$$\begin{aligned} v_1(x) &= \int v_1'(x)\,dx = \int -\frac{1}{2}\frac{1}{1+e^x}\,dx = -\frac{1}{2}x + \frac{1}{2}\ln(1+e^x) \\ v_2(x) &= \int v_2'(x)\,dx = \int \frac{1}{2}\frac{e^{-2x}}{1+e^x}\,dx = -\frac{1}{4}e^{-2x} + \frac{1}{2}e^{-x} - \frac{1}{2}\ln(1+e^{-x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= c_1e^x + c_2e^{3x} + e^x\left(-\frac{1}{2}x + \frac{1}{2}\ln(1+e^x)\right) + e^{3x}\left(-\frac{1}{4}e^{-2x} + \frac{1}{2}e^{-x} - \frac{1}{2}\ln(1+e^{-x})\right) \\ &= \left(c_1 - \frac{1}{4}\right)e^x + c_2e^{3x} - \frac{1}{2}xe^x + \frac{1}{2}e^x\ln\left(1+e^x\right) - \frac{1}{2}e^{3x}\ln\left(1+e^{-x}\right) + \frac{1}{2}e^{2x} \end{aligned}$$

□