MATH 046 - Spring 2018

Worked Problems - Chapter 4

NOTE: The sign and/or constants being multiplied in front of the arbitrary constant C don't matter, as multiplying a constant by a minus sign or another constant is still just a constant. So it is fine if your constant looks different, as long as the rest of the solution is correct.

1. Solve the given differential equation

$$(t+1)\ dt - \frac{1}{y^2}\ dy = 0$$

***Solution*:** The ODE is separable, so we do the separation, and then solve

$$\begin{aligned}(t+1)\ dt - \frac{1}{y^2}\ dy &= 0\\ -\frac{1}{y^2}\ dy &= -(t+1)\ dt\\ \int \frac{1}{y^2}\ dy &= \int (t+1)\ dt\\ -\frac{1}{y} &= \frac{1}{2}t^2 + t + C\\ y(t) &= -\frac{1}{\frac{1}{2}t^2 + t + C}\end{aligned}$$

□

2. Solve the given differential equation

$$dx - \frac{1}{y^2+1}\ dy = 0$$

***Solution*:** The ODE is separable, so we do the separation, and then solve

$$\begin{aligned}dx - \frac{1}{y^2+1}\ dy &= 0\\ \frac{1}{y^2+1}\ dy &= \ dx\\ \int \frac{1}{y^2+1}\ dy &= \int\ dx \qquad \text{use } \int \frac{1}{u^2+a^2}\ du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)\\ \tan^{-1}(y) &= x + C\\ y(x) &= \tan(x+C)\end{aligned}$$

□

3. Solve the given differential equation

$$dx - \frac{1}{y^2-6y+13}\,dy = 0$$

***Solution*:** The ODE is separable, so we do the separation, and then solve

$$\begin{aligned}
dx - \frac{1}{y^2-6y+13}\,dy &= 0\\
\frac{1}{y^2-6y+13}\,dy &= dx\\
\frac{1}{y^2-6y+9+4}\,dy &= dx\\
\frac{1}{(y-3)^2+4}\,dy &= dx\\
\int \frac{1}{(y-3)^2+4}\,dy &= \int dx \qquad \text{use } \int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)\\
\frac{1}{2}\tan^{-1}\left(\frac{y-3}{2}\right) &= x+C\\
\tan^{-1}\left(\frac{y-3}{2}\right) &= 2x+2C\\
y-3 &= 2\tan(2x+2C)\\
y(x) &= 3+2\tan(2x+2C)
\end{aligned}$$

□

4. Solve the given differential equation

$$\frac{dy}{dt} = 3+5y$$

***Solution*:** The ODE is separable, so we do the separation, and then solve

$$\begin{aligned}
\frac{dy}{dt} &= 3+5y\\
\frac{1}{3+5y}\,dy &= dt\\
\int \frac{1}{3+5y}\,dy &= \int dt\\
\frac{1}{5}\ln|3+5y| &= t+C\\
\ln|3+5y| &= 5(t+C)\\
|3+5y| &= e^{5(t+C)}\\
3+5y &= \pm e^{5(t+C)}\\
y(t) &= -\frac{3}{5} \pm \frac{1}{5}e^{5(t+C)}\\
y(t) &= -\frac{3}{5} \pm \frac{1}{5}e^{5C}e^{5t}
\end{aligned}$$

□

5. Solve the given initial value problem

$$\sin(x)\ dx + y\ dy = 0 \qquad y(0) = -2$$

***Solution*:** The ODE is separable, so we do the separation, and then solve

$$\begin{aligned}
\sin(x)\ dx + y\ dy &= 0 \\
y\ dy &= -\sin(x)\ dx \\
\int y\ dy &= \int -\sin(x)\ dx \\
\frac{y^2}{2} &= \cos(x) + C \\
(y(x))^2 &= 2\cos(x) + 2C \qquad \text{general solution} \\
(y(0))^2 &= 2\cos(0) + 2C \\
4 &= 2 + 2C \\
C &= 1 \\
\Rightarrow \quad y(x)^2 &= 2 + 2\cos(x) \qquad \text{solution to IVP}
\end{aligned}$$

□

6. Solve the given initial value problem

$$\frac{dy}{dx} = \frac{x^2 y - y}{y+1} \qquad y(3) = -1$$

***Solution*:** The ODE is separable, so we do the separation, and then solve

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2 y - y}{y+1} \\
(y+1)\ dy &= (y(x^2-1))\ dx \\
\int \left(1 + \frac{1}{y}\right)\ dy &= \int x^2 - 1\ dx \\
y + \ln|y| &= \frac{1}{3}x^3 - x + C \\
y(x) + \ln|y(x)| &= \frac{1}{3}x^3 - x + C \qquad \text{general solution} \\
y(3) + \ln|y(3)| &= \frac{1}{3}3^3 - 3 + C \\
-1 + \ln|-1| &= 9 - 3 + C \\
C &= -7 \\
\Rightarrow \quad y(x) + \ln|y(x)| &= \frac{1}{3}x^3 - x - 7 \qquad \text{solution to IVP}
\end{aligned}$$

□

7. The population of a city is known to grow at a rate proportional to the number of people presently living in this city. Suppose the annual rate is 3% and the population is 500,000 in the year 2018. Can you find out the population at the year 1998?

***Solution*:** We set up the population ODE from previous homework

$$\frac{dP}{dt} = kP(t)$$

where k is the growth constant. Now we just solve this separable ODE

$$\begin{aligned} \frac{dP}{dt} &= kP(t) \\ \frac{1}{P}\ dP &= k\ dt \\ \int \left(\frac{1}{P}\right)\ dP &= \int k\ dt \\ \ln|P| &= kt + C \qquad \text{Population must be positive, so we drop } |\cdot| \\ P &= e^{kt+C} = \widetilde{C}e^{kt} \qquad \text{general solution} \\ \Rightarrow \quad P(t) &= 500000e^{0.03t} \end{aligned}$$

where we have assumed in the last line that at $t = 0$ (which will be the year 2018), the starting population will be $\widetilde{C} = 500000$ and the rate constant is $k = 0.03$. Now, the year 1998 is 20 years before 2018, so if $t = 0$ is 2018, then $t = -20$ is 1998. Therefore, we seek $P(-20)$. Thus,

$$P(t) = 500000e^{0.03t} \quad \Rightarrow \quad P(-20) = 500000e^{0.03(-20)} = 500000e^{-0.6} \approx 274405$$

$\square$