MATH 046 - Spring 2018

Worked Problems - Chapter 5

NOTE: The sign and/or constants being multiplied in front of the arbitrary constant C don't matter, as multiplying a constant by a minus sign or another constant is still just a constant. So it is fine if your constant looks different, as long as the rest of the solution is correct.

1. Solve the given differential equation

$$(y + 2xy^3)\ dx + (1 + 3x^2y^2 + x)\ dy = 0$$

***Solution*:** First we test if the ODE is exact:

$$\frac{\partial M}{\partial y} = 1 + 6xy^2$$
$$\frac{\partial N}{\partial x} = 1 + 6xy^2$$

The equation is exact, so we can solve the ODE via the method in the book

$$\frac{\partial g}{\partial x} = y + 2xy^3 \ \Rightarrow\ g(x,y) = xy + x^2y^3 + h(y)$$
$$\Rightarrow\ \frac{\partial g}{\partial y} = x + 3x^2y^2 + h'(y)$$
$$\Rightarrow\ 1 + 3x^2y^2 + x = x + 3x^2y^3 + h'(y)$$
$$\Rightarrow\ h'(y) = 1\ \Rightarrow\ h(y) = y$$
$$\Rightarrow\ xy + x^2y^3 + y = C$$

□

2. Solve the given differential equation

$$y\ dx + x\ dy = 0$$

***Solution*:** First we test if the ODE is exact:

$$\frac{\partial M}{\partial y} = 1$$
$$\frac{\partial N}{\partial x} = 1$$

The equation is exact. There is a lot of ways to solve this ODE. To get the answer in the key, just use the chart in the textbook, which has this ODE explicitly given with its integrating factor. We deduce that the solution is

$$xy = C$$

□

3. Solve the given differential equation

$$\frac{ty-1}{t^2y}\,dt - \frac{1}{ty^2}\,dy = 0$$

***Solution*:** First we test if the ODE is exact:

$$\frac{\partial M}{\partial y} = \frac{1}{t^2y^2}$$
$$\frac{\partial N}{\partial t} = \frac{1}{t^2y^2}$$

The equation is exact, so we can solve the ODE via the method in the book

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \frac{ty-1}{t^2y} = \frac{1}{t} - \frac{1}{t^2y} \;\Rightarrow\; g(t,y) = \ln(|t|) + \frac{1}{ty} + h(y)\\
\Rightarrow\; \frac{\partial g}{\partial y} &= -\frac{1}{ty^2} + h'(y)\\
\Rightarrow\; -\frac{1}{ty^2} &= -\frac{1}{ty^2} + h'(y)\\
\Rightarrow\; h'(y) = 0 &\;\Rightarrow\; h(y) = C\\
\Rightarrow\; \ln(|t|) + \frac{1}{ty} &+ C = C_1\\
\Rightarrow\; \ln(|kt|) &= -\frac{1}{ty}\\
y(t) &= -\frac{1}{t\ln(|kt|)}
\end{aligned}$$

□

4. Solve the given differential equation

$$(t^2 - x)\,dt - t\,dx = 0$$

***Solution*:** First we test if the ODE is exact:

$$\frac{\partial M}{\partial x} = -1$$
$$\frac{\partial N}{\partial t} = -1$$

The equation is exact, so we can solve the ODE via the method in the book

$$\begin{aligned}
\frac{\partial g}{\partial t} &= t^2 - x \;\Rightarrow\; g(t,x) = \frac{1}{3}t^3 - tx + h(x)\\
\Rightarrow\; \frac{\partial g}{\partial x} &= -t + h'(x)\\
\Rightarrow\; -t &= -t + h'(x)\\
\Rightarrow\; h'(x) = 0 &\;\Rightarrow\; h(x) = C_1\\
\Rightarrow\; \frac{1}{3}t^3 &- tx + C_1 = C_2\\
\Rightarrow\; -tx &= -\frac{1}{3}t^3 + C\\
x(t) &= \frac{1}{3}t^2 - \frac{C}{t}
\end{aligned}$$

□

5. Solve the given differential equation

$$2xe^{2t}\ dt + (1 + e^{2t})\ dx = 0$$

***Solution*:** First we test if the ODE is exact:

$$\frac{\partial M}{\partial x} = 2e^{2t}$$
$$\frac{\partial N}{\partial t} = 2e^{2t}$$

The equation is exact, so we can solve the ODE via the method in the book

$$\begin{aligned}
\frac{\partial g}{\partial t} &= 2xe^{2t} \;\Rightarrow\; g(t,x) = xe^{2t} + h(x) \\
\Rightarrow\; \frac{\partial g}{\partial x} &= e^{2t} + h'(x) \\
\Rightarrow\; 1 + e^{2t} &= e^{2t} + h'(x) \\
\Rightarrow\; h'(x) = 1 &\;\Rightarrow\; h(x) = x \\
\Rightarrow\; xe^{2t} + x &= C \\
\Rightarrow\; x(1 + e^{2t}) &= C \\
x(t) &= \frac{C}{1 + e^{2t}}
\end{aligned}$$

□

6. Solve the given differential equation

$$(\cos(x) + x\cos(t))\ dt + (\sin(t) - t\sin(x))\ dx = 0$$

***Solution*:** First we test if the ODE is exact:

$$\frac{\partial M}{\partial x} = \cos(t) - \sin(x)$$
$$\frac{\partial N}{\partial t} = \cos(t) - \sin(x)$$

The equation is exact, so we can solve the ODE via the method in the book

$$\begin{aligned}
\frac{\partial g}{\partial t} &= \cos(x) + x\cos(t) \;\Rightarrow\; g(t,x) = t\cos(x) + x\sin(t) + h(x) \\
\Rightarrow\; \frac{\partial g}{\partial x} &= \sin(t) - t\sin(x) + h'(x) \\
\Rightarrow\; \sin(t) - t\sin(x) &= \sin(t) - t\sin(x) + h'(x) \\
\Rightarrow\; h'(x) = 0 &\;\Rightarrow\; h(x) = C_1 \\
\Rightarrow\; t\cos(x) + x\sin(t) + C_1 &= C_2 \\
t\cos(x) + x\sin(t) &= C
\end{aligned}$$

□

7. Solve the given initial value problem

$$(y + 2xy^3)\ dx + (1 + 3x^2y^2 + x)\ dy = 0 \qquad y(1) = -5$$

***Solution*:** From problem number 1, we already have the solution. We just apply the initial condition

$$\begin{aligned}
\Rightarrow\ xy + x^2y^3 + y &= C \\
\Rightarrow\ (1)y(1) + (1^2)y(1)^3 + y(1) &= C \\
\Rightarrow\ -5 + (-5)^3 - 5 &= C \\
\Rightarrow\ -135 &= C \\
xy + x^2y^3 + y &= -135
\end{aligned}$$

□

8. Solve the given initial value problem

$$(t^2 - x)\ dt - t\ dx = 0 \qquad x(1) = 5$$

***Solution*:** From problem number 4, we already have the solution. We just apply the initial condition

$$\begin{aligned}
\Rightarrow\ x(t) &= \frac{1}{3}t^2 - \frac{C}{t} \\
\Rightarrow\ x(1) &= \frac{1}{3}(1^2) - \frac{C}{1} \\
\Rightarrow\ 5 &= \frac{1}{3} - C \\
\Rightarrow\ -\frac{14}{3} &= C \\
x(t) &= \frac{1}{3}t^2 + \frac{14}{3}\left(\frac{1}{t}\right)
\end{aligned}$$

□