MATH 046 - Spring 2018

Worked Problems - Chapter 7

1. A body of mass 3 slugs is dropped from a height of 500 ft in a with zero velocity. Assuming no air resistance, find
(a) an expression for the velocity of the body at any time t
(b) an expression for the position of the body at any time t.

***Solution*:** (a) Use the example 7.11 as a guide. The "positive" orientation is given to be pointing downward, which is in the same direction as the motion, so $v(t)$ will be positive. It states that the motion is governed by the ODE

$$F = ma \quad \Rightarrow m\frac{dv}{dt} = mg \quad \Rightarrow \quad \frac{dv}{dt} = g$$

where g is a constant. This is a separable ODE, which we solve

$$\begin{aligned} \frac{dv}{dt} &= g \\ dv &= g\ dt \\ \int dv &= \int g\ dt \\ v &= gt + C \\ v(t) &= gt + C \qquad \text{general solution} \\ v(0) &= g(0) + C \\ C &= 0 \\ v(t) &= gt \qquad \text{solution with zero initial velocity } v(0) = 0 \end{aligned}$$

(b) To find the position function, we integrate the velocity function

$$\begin{aligned} v(t) &= gt \\ \int v(t)\ dt &= \int gt\ dt \\ s(t) &= \frac{1}{2}gt^2 + C \qquad \text{general solution} \\ s(0) &= \frac{1}{2}g(0^2) + C \\ C &= 0 \\ s(t) &= \frac{1}{2}gt^2 \qquad \text{solution with initial position at } s(0) = 0 \end{aligned}$$

□

2. Redo Problem 1 above, assuming there is air resistance which creates a force on the body equal to $-2v$ lbs.

***Solution*:** (a) Use the example 7.11 as a guide. The "positive" orientation is given to be pointing downward, which is in the same direction as the motion, so $v(t)$ will be positive. The air resistance acts opposite the direction of motion, that is why the air resistance is $-2v$ lbs. The motion is governed by the ODE

$$F = ma \quad \Rightarrow m\frac{dv}{dt} = mg - kv \quad \Rightarrow \quad \frac{dv}{dt} = g - \frac{k}{m}v \quad \Rightarrow \quad \frac{dv}{dt} + \frac{k}{m}v = g$$

Here, our $-kv = -2v \quad \Rightarrow \quad kv = 2v$ from the problem statement, as well as $m = 3$. Therefore we solve the ODE

$$\begin{aligned}
\frac{dv}{dt} + \frac{2}{3}v &= g \\
\frac{dv}{dt}e^{\frac{2}{3}t} + \frac{2}{3}e^{\frac{2}{3}t}v &= ge^{\frac{2}{3}t} \\
\frac{d}{dt}\left[ve^{\frac{2}{3}t}\right] &= ge^{\frac{2}{3}t} \\
\int \frac{d}{dt}\left[ve^{\frac{2}{3}t}\right]\, dt &= \int ge^{\frac{2}{3}t}\, dt \\
ve^{\frac{2}{3}t} &= C + \frac{3}{2}ge^{\frac{2}{3}t} \\
v(t) &= Ce^{-\frac{2}{3}t} + \frac{3}{2}g \qquad \text{general solution} \\
v(0) &= Ce^{-\frac{2}{3}0} + \frac{3}{2}g \\
C &= -\frac{3}{2}g \\
v(t) &= \frac{3}{2}g - \frac{3}{2}ge^{-\frac{2}{3}t} \qquad \text{solution with zero initial velocity } v(0) = 0
\end{aligned}$$

(b) To find the position function, we integrate the velocity function

$$\begin{aligned}
v(t) &= \frac{3}{2}g - \frac{3}{2}ge^{-\frac{2}{3}t} \\
\int v(t)\, dt &= \int \frac{3}{2}g - \frac{3}{2}ge^{-\frac{2}{3}t}\, dt \\
s(t) &= \frac{3}{2}gt + \frac{9}{4}ge^{-\frac{2}{3}t} + C \qquad \text{general solution} \\
s(0) &= \frac{3}{2}g(0) + \frac{9}{4}ge^{-\frac{2}{3}0} + C \\
C &= -\frac{9}{4}g \\
s(t) &= \frac{9}{4}ge^{-\frac{2}{3}t} + \frac{3}{2}gt - \frac{9}{4}g \qquad \text{solution with initial position at } s(0) = 0
\end{aligned}$$

□

3. Suppose you opened a savings account with the annual interest rate 2% and deposited \$100,000 at the time you opened it. Suppose the interest compounds continuously in time and you deposit \$10,000 into this account annually. How much money do you have in this account after 10 years?

***Solution*:** First we must set up the differential equation describing this situation. Let $P = 100000$ be the principal (initial amount in account), $r = 0.02$ the interest rate, and S be the amount added annually. The rate at which money accumulates from interest is just rA if $A(t)$ represents the money in the account at time t, since we compound continuously. Since we add 10,000 for every year, this is equivalent to

$$\begin{aligned}
\frac{dA}{dt} &= rA + 10000 \\
\frac{dA}{dt}e^{-rt} - re^{-rt}A &= 10000e^{-rt} \\
\frac{d}{dt}\left[Ae^{-rt}\right] &= 10000e^{-rt} \\
Ae^{-rt} &= C - \frac{10000}{r}e^{-rt} \\
A(t) &= Ce^{rt} - \frac{10000}{r} \\
\text{using } A(0) &= 100000 \quad \Rightarrow \quad C = 100000 + \frac{10000}{r} \\
A(t) &= 100000e^{rt} + \frac{10000}{r}\left(e^{rt} - 1\right) \\
A(10) &= 100000e^{0.02(10)} + \frac{10000}{0.02}\left(e^{0.02(10)} - 1\right)
\end{aligned}$$

We can recover this in an alternate way. We can express the amount if interest is compounded n times, where t is overall amount of time. Let S be the amount we add each period. We can write how much we money have at each n by

$$\begin{aligned}
A_n(t) &= P\left(1 + \frac{r}{n}\right)^{nt} + \frac{S\left(1 + \frac{r}{n}\right)^{nt} - S}{r} \\
\lim_{n\to\infty} A_n(t) &= \lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} + \frac{S\left(1 + \frac{r}{n}\right)^{nt} - S}{r} \\
A(t) &= Pe^{rt} + \frac{S}{r}\left(e^{rt} - 1\right)
\end{aligned}$$

where in the first line, we have the amount after the interest, plus the second term which is the interest plus the amount we are adding, S, subtracted by the added amount, S, over the interest rate. Taking the limit tells us we will compound continuously, and we get the same solution as the ODE method. □