**MATH 046 - Spring 2018**

Worked Problems - Chapter 8

**1.** Which of the following ODEs are linear, linear and homogeneous, or is linear and has constant coefficients?

$$\begin{aligned}
&\text{(a) } y'' + xy' + 2y = 0 \\
&\text{(g) } y'' + yy' = x^2 \\
&\text{(j) } y' + x\sin(y) = x \\
&\text{(k) } y'' + e^y = 0 \\
&\text{(l) } y'' + e^x = 0
\end{aligned}$$

***Solution*:** The ODEs (a) and (l) are linear since they do not contain any functions of $y, y'$, or $y''$, thus (g), (j), and (k) are all nonlinear ODEs. Only (a) is linear and homogeneous, as (l) can be rewritten as $y'' = -e^x$. Only (l) is linear and has constant coefficients, as (a) has the $xy'$ term. □

**2.** Find the Wronskian of the given set of functions and determine whether the set of functions is linear independent. Use the result, construct the general solution of $y'' - 4y = 0$.

$$\left\{e^{2x}, e^{-2x}\right\}$$

***Solution*:** We compute the Wronskian directly

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = e^{2x}(-2e^{-2x}) - (e^{-2x})(2e^{2x}) = -4$$

So since the Wronskian is non-zero, the functions are linearly independent. Since the functions are linearly independent, and both satisfy the ODE $y'' - 4y = 0$, therefore the linear combination of solutions is also solution. The general solution is then

$$y(x) = c_1 e^{2x} + c_2 e^{-2x}$$

□

**3.** Find the Wronskian of the given set of functions and determine whether the set of functions is linear independent. Use the result, construct the general solution of $y'' - 5y' + 6y = 0$.

$$\left\{e^{2x}, e^{3x}\right\}$$

***Solution*:** We compute the Wronskian directly

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{2x}(3e^{3x}) - (e^{3x})(2e^{2x}) = e^{5x}$$

So since the Wronskian is non-zero, the functions are linearly independent. Since the functions are linearly independent, and both satisfy the ODE $y'' - 5y' + 6y = 0$, therefore the linear combination of solutions is also solution. The general solution is then

$$y(x) = c_1 e^{2x} + c_2 e^{3x}$$

□

**4.** What can one say about the general solution of $y'' + 16y = 0$ if two particular solutions are known to be $y_1 = \sin(4x)$ and $y_2 = \cos(4x)$?

***Solution*:** We compute the Wronskian directly

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \sin(4x) & \cos(4x) \\ 4\cos(4x) & -4\sin(4x) \end{vmatrix} = -4\sin^2(4x) - 4\cos^2(4x) = -4$$

So since the Wronskian is non-zero, the functions are linearly independent. Since the functions are linearly independent, and both satisfy the ODE $y'' + 16y = 0$, therefore the linear combination of solutions is also solution. The general solution is then

$$y(x) = c_1 \sin(4x) + c_2 \cos(4x)$$

□

**5.** What can one say about the general solution of $y'' - 8y' = 0$ if two particular solutions are known to be $y_1 = e^{8x}$ and $y_2 = 1$?

***Solution*:** We compute the Wronskian directly

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} e^{8x} & 1 \\ 8e^{8x} & 0 \end{vmatrix} = -8e^{8x}$$

So since the Wronskian is non-zero, the functions are linearly independent. Since the functions are linearly independent, and both satisfy the ODE $y'' - 8y' = 0$, therefore the linear combination of solutions is also solution. The general solution is then

$$y(x) = c_1 e^{8x} + c_2$$

□

**6.** Find the general solution of $y'' + y = x^2$, if one solution is $y = x^2 - 2$, and if two solutions of $y'' + y = 0$ are $\sin(x)$ and $\cos(x)$.

***Solution*:** We compute the Wronskian directly for the two solutions of the homogeneous equation

$$W = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -\sin^2(x) - \cos^2(x) = -1$$

So since the Wronskian is non-zero, the functions are linearly independent. Since the functions are linearly independent, and both satisfy the ODE $y'' + y = 0$, therefore the linear combination of solutions is also solution. Since $y = x^2 - 2$ is a particular solution to the ODE $y'' + y = x^2$, then the general solution is a linear combination of all solutions

$$y(x) = c_1 \sin(x) + c_2 \cos(x) + x^2 - 2$$

□