28. (Overdamped) If $x_0 = 0$, deduce from Problem 27 that

$$x(t) = \frac{v_0}{\gamma} e^{-pt} \sinh \gamma t.$$

29. (Overdamped) Prove that in this case the mass can pass through its equilibrium position $x = 0$ at most once.

30. (Underdamped) Show that in this case

$$x(t) = e^{-pt} \left(x_0 \cos \omega_1 t + \frac{v_0 + p x_0}{\omega_1} \sin \omega_1 t \right).$$

31. (Underdamped) If the damping constant c is small in comparison with $\sqrt{8mk}$, apply the binomial series to show that

$$\omega_1 \approx \omega_0 \left(1 - \frac{c^2}{8mk} \right).$$

32. (Underdamped) Show that the local maxima and minima of

$$x(t) = C e^{-pt} \cos(\omega_1 t - \alpha)$$

occur where

$$\tan(\omega_1 t - \alpha) = -\frac{p}{\omega_1}.$$

Conclude that $t_2 - t_1 = 2\pi/\omega_1$ if two consecutive maxima occur at times t_1 and t_2.

33. (Underdamped) Let x_1 and x_2 be two consecutive local maximum values of $x(t)$. Deduce from the result of Problem 32 that

$$\ln \frac{x_1}{x_2} = \frac{2\pi p}{\omega_1}.$$

The constant $\Delta = 2\pi p/\omega_1$ is called the **logarithmic decrement** of the oscillation. Note also that $c = m\omega_1 \Delta/\pi$ because $p = c/(2m)$.

***Note**: The result of Problem 33 provides an accurate method for measuring the* viscosity *of a fluid, which is an important parameter in fluid dynamics but is not easy to measure directly. According to Stokes's drag law, a spherical body of radius a moving at a (relatively slow) speed through a fluid of viscosity μ experiences a resistive force $F_R = 6\pi\mu a v$. Thus if a spherical mass on a spring is immersed in the fluid and set in motion, this drag resistance damps its oscillations with damping constant $c = 6\pi a\mu$. The frequency ω_1 and logarithmic decrement Δ of the oscillations can be measured by direct observation. The final formula in Problem 33 then gives c and hence the viscosity of the fluid.*

34. (Underdamped) A body weighing 100 lb (mass m = 3.125 slugs in fps units) is oscillating attached to a spring and a dashpot. Its first two maximum displacements of 6.73 in. and 1.46 in. are observed to occur at times 0.34 s and 1.17 s, respectively. Compute the damping constant (in pound-seconds per foot) and spring constant (in pounds per foot).

Differential Equations and Determinism

Given a mass m, a dashpot constant c, and a spring constant k, Theorem 2 of Section 2.1 implies that the equation

$$mx'' + cx' + kx = 0 \tag{26}$$

has a unique *solution for $t \geqq 0$ satisfying given initial conditions $x(0) = x_0$, $x'(0) = v_0$. Thus the future motion of an ideal mass–spring–dashpot system is completely determined by the differential equation and the initial conditions. Of course in a real physical system it is impossible to measure the parameters m, c, and k* precisely. *Problems 35 through 38 explore the resulting uncertainty in predicting the future behavior of a physical system.*

35. Suppose that $m = 1$, $c = 2$, and $k = 1$ in Eq. (26). Show that the solution with $x(0) = 0$ and $x'(0) = 1$ is

$$x_1(t) = te^{-t}.$$

36. Suppose that $m = 1$ and $c = 2$ but $k = 1 - 10^{-2n}$. Show that the solution of Eq. (26) with $x(0) = 0$ and $x'(0) = 1$ is

$$x_2(t) = 10^n e^{-t} \sinh 10^{-n} t.$$

37. Suppose that $m = 1$ and $c = 2$ but that $k = 1 + 10^{-2n}$. Show that the solution of Eq. (26) with $x(0) = 0$ and $x'(0) = 1$ is

$$x_3(t) = 10^n e^{-t} \sin 10^{-n} t.$$

38. Whereas the graphs of $x_1(t)$ and $x_2(t)$ resemble those shown in Figs. 2.4.7 and 2.4.8, the graph of $x_3(t)$ exhibits damped oscillations like those illustrated in Fig. 2.4.9, but with a very long pseudoperiod. Nevertheless, show that for each fixed $t > 0$ it is true that

$$\lim_{n\to\infty} x_2(t) = \lim_{n\to\infty} x_3(t) = x_1(t).$$

Conclude that *on a given finite time interval* the three solutions are in "practical" agreement if n is sufficiently large.

2.5 Nonhomogeneous Equations and Undetermined Coefficients

We learned in Section 2.3 how to solve homogeneous linear equations with constant coefficients, but we saw in Section 2.4 that an external force in a simple mechanical system contributes a nonhomogeneous term to its differential equation. The general nonhomogeneous nth-order linear equation with constant coefficients has the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = f(x). \tag{1}$$

By Theorem 5 of Section 2.2, a general solution of Eq. (1) has the form

$$y = y_c + y_p \tag{2}$$

where the complementary function $y_c(x)$ is a general solution of the associated homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{3}$$

and $y_p(x)$ is a particular solution of Eq. (1). Thus our remaining task is to find y_p.

The **method of undetermined coefficients** is a straightforward way of doing this when the given function $f(x)$ in Eq. (1) is sufficiently simple that we can make an intelligent guess as to the general form of y_p. For example, suppose that $f(x)$ is a polynomial of degree m. Then, because the derivatives of a polynomial are themselves polynomials of lower degree, it is reasonable to suspect a particular solution

$$y_p(x) = A_m x^m + A_{m-1} x^{m-1} + \cdots + A_1 x + A_0$$

that is also a polynomial of degree m, but with as yet undetermined coefficients. We may, therefore, substitute this expression for y_p into Eq. (1), and then—by equating coefficients of like powers of x on the two sides of the resulting equation—attempt to determine the coefficients $A_0, A_1, \ldots, A_m$ so that y_p will, indeed, be a particular solution of Eq. (1).

Similarly, suppose that

$$f(x) = a \cos kx + b \sin kx.$$

Then it is reasonable to expect a particular solution of the same form:

$$y_p(x) = A \cos kx + B \sin kx,$$

a linear combination with undetermined coefficients A and B. The reason is that any derivative of such a linear combination of $\cos kx$ and $\sin kx$ has the same form. We may therefore substitute this form of y_p in Eq. (1), and then—by equating coefficients of $\cos kx$ and $\sin kx$ on both sides of the resulting equation—attempt to determine the coefficients A and B so that y_p will, indeed, be a particular solution.

It turns out that this approach does succeed whenever all the derivatives of $f(x)$ have the same form as $f(x)$ itself. Before describing the method in full generality, we illustrate it with several preliminary examples.

Example 1 Find a particular solution of $y'' + 3y' + 4y = 3x + 2$.

Solution Here $f(x) = 3x + 2$ is a polynomial of degree 1, so our guess is that

$$y_p(x) = Ax + B.$$

Then $y_p' = A$ and $y_p'' = 0$, so y_p will satisfy the differential equation provided that

$$(0) + 3(A) + 4(Ax + B) = 3x + 2,$$

that is,

$$(4A)x + (3A + 4B) = 3x + 2$$

for all x. This will be true if the x-terms and constant terms on the two sides of this equation agree. It therefore suffices for A and B to satisfy the two linear equations $4A = 3$ and $3A + 4B = 2$ that we readily solve for $A = \frac{3}{4}$ and $B = -\frac{1}{16}$. Thus we have found the particular solution

$$y_p(x) = \tfrac{3}{4}x - \tfrac{1}{16}. \qquad \blacksquare$$

Example 2 Find a particular solution of $y'' - 4y = 2e^{3x}$.

Solution Any derivative of e^{3x} is a constant multiple of e^{3x}, so it is reasonable to try

$$y_p(x) = Ae^{3x}.$$

Then $y_p'' = 9Ae^{3x}$, so the given differential equation will be satisfied provided that

$$9Ae^{3x} - 4(Ae^{3x}) = 2e^{3x};$$

that is, $5A = 2$, so that $A = \frac{2}{5}$. Thus our particular solution is $y_p(x) = \frac{2}{5}e^{3x}$. $\blacksquare$

Example 3 Find a particular solution of $3y'' + y' - 2y = 2\cos x$.

Solution A first guess might be $y_p(x) = A\cos x$, but the presence of y' on the left-hand side signals that we probably need a term involving $\sin x$ as well. So we try

$$\begin{aligned} y_p(x) &= A\cos x + B\sin x; \\ y_p'(x) &= -A\sin x + B\cos x, \\ y_p''(x) &= -A\cos x - B\sin x. \end{aligned}$$

Then substitution of y_p and its derivatives into the given differential equation yields

$$3(-A\cos x - B\sin x) + (-A\sin x + B\cos x) - 2(A\cos x + B\sin x) = 2\cos x,$$

that is (collecting coefficients on the left),

$$(-5A + B)\cos x + (-A - 5B)\sin x = 2\cos x.$$

This will be true for all x provided that the cosine and sine terms on the two sides of this equation agree. It therefore suffices for A and B to satisfy the two linear equations

$$\begin{aligned} -5A + B &= 2, \\ -A - 5B &= 0 \end{aligned}$$

with readily found solution $A = -\frac{5}{13}$, $B = \frac{1}{13}$. Hence a particular solution is

$$y_p(x) = -\tfrac{5}{13}\cos x + \tfrac{1}{13}\sin x. \qquad \blacksquare$$

The following example, which superficially resembles Example 2, indicates that the method of undetermined coefficients is not always quite so simple as we have made it appear.

Example 4 Find a particular solution of $y'' - 4y = 2e^{2x}$.

Solution If we try $y_p(x) = Ae^{2x}$, we find that

$$y_p'' - 4y_p = 4Ae^{2x} - 4Ae^{2x} = 0 \neq 2e^{2x}.$$

Thus, no matter how A is chosen, Ae^{2x} cannot satisfy the given nonhomogeneous equation. In fact, the preceding computation shows that Ae^{2x} satisfies instead the associated *homogeneous* equation. Therefore, we should begin with a trial function $y_p(x)$ whose derivative involves both e^{2x} *and something else* that can cancel upon substitution into the differential equation to leave the e^{2x} term that we need. A reasonable guess is

$$y_p(x) = Axe^{2x},$$

for which

$$y_p'(x) = Ae^{2x} + 2Axe^{2x} \quad \text{and} \quad y_p''(x) = 4Ae^{2x} + 4Axe^{2x}.$$

Substitution into the original differential equation yields

$$(4Ae^{2x} + 4Axe^{2x}) - 4(Axe^{2x}) = 2e^{2x}.$$

The terms involving xe^{2x} obligingly cancel, leaving only $4Ae^{2x} = 2e^{2x}$, so that $A = \frac{1}{2}$. Consequently, a particular solution is

$$y_p(x) = \tfrac{1}{2}xe^{2x}. \qquad \blacksquare$$

The General Approach

Our initial difficulty in Example 4 resulted from the fact that $f(x) = 2e^{2x}$ satisfies the associated homogeneous equation. Rule 1, given shortly, tells what to do when we do not have this difficulty, and Rule 2 tells what to do when we do have it.

The method of undetermined coefficients applies whenever the function $f(x)$ in Eq. (1) is a linear combination of (finite) products of functions of the following three types:

1. A polynomial in x;
2. An exponential function e^{rx}; (4)
3. $\cos kx$ or $\sin kx$.

Any such function—for example,

$$f(x) = (3 - 4x^2)e^{5x} - 4x^3 \cos 10x,$$

has the crucial property that only *finitely* many linearly independent functions appear as terms (summands) in $f(x)$ and its derivatives of all orders. In Rules 1 and 2 we assume that $Ly = f(x)$ is a nonhomogeneous linear equation with constant coefficients and that $f(x)$ is a function of this kind.

RULE 1 Method of Undetermined Coefficients

Suppose that no term appearing either in $f(x)$ or in any of its derivatives satisfies the associated homogeneous equation $Ly = 0$. Then take as a trial solution for y_p a linear combination of all linearly independent such terms and their derivatives. Then determine the coefficients by substitution of this trial solution into the nonhomogeneous equation $Ly = f(x)$.

Note that this rule is not a theorem requiring proof; it is merely a procedure to be followed in searching for a particular solution y_p. If we succeed in finding y_p, then nothing more need be said. (It can be proved, however, that this procedure will always succeed under the conditions specified here.)

In practice we check the supposition made in Rule 1 by first using the characteristic equation to find the complementary function y_c, and then write a list of all the terms appearing in $f(x)$ and its successive derivatives. If none of the terms in this list duplicates a term in y_c, then we proceed with Rule 1.

Example 5 Find a particular solution of

$$y'' + 4y = 3x^3. \tag{5}$$

Solution The (familiar) complementary solution of Eq. (5) is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x.$$

The function $f(x) = 3x^3$ and its derivatives are constant multiples of the linearly independent functions x^3, x^2, x, and 1. Because none of these appears in y_c, we try

$$\begin{aligned} y_p &= Ax^3 + Bx^2 + Cx + D, \\ y_p' &= 3Ax^2 + 2Bx + C, \\ y_p'' &= 6Ax + 2B. \end{aligned}$$

Substitution in Eq. (5) gives

$$\begin{aligned} y_p'' + 4y_p &= (6Ax + 2B) + 4(Ax^3 + Bx^2 + Cx + D) \\ &= 4Ax^3 + 4Bx^2 + (6A + 4C)x + (2B + D) = 3x^3. \end{aligned}$$

We equate coefficients of like powers of x in the last equation to get

$$\begin{aligned} 4A &= 3, & 4B &= 0, \\ 6A + 4C &= 0, & 2B + D &= 0 \end{aligned}$$

with solution $A = \frac{3}{4}$, $B = 0$, $C = -\frac{9}{8}$, and $D = 0$. Hence a particular solution of Eq. (5) is

$$y_p(x) = \tfrac{3}{4}x^3 - \tfrac{9}{8}x.$$

■

Example 6 Solve the initial value problem

$$\begin{aligned} &y'' - 3y' + 2y = 3e^{-x} - 10\cos 3x; \\ &y(0) = 1, \quad y'(0) = 2. \end{aligned} \tag{6}$$

Solution The characteristic equation $r^2 - 3r + 2 = 0$ has roots $r = 1$ and $r = 2$, so the complementary function is

$$y_c(x) = c_1 e^x + c_2 e^{2x}.$$

The terms involved in $f(x) = 3e^{-x} - 10\cos 3x$ and its derivatives are e^{-x}, $\cos 3x$, and $\sin 3x$. Because none of these appears in y_c, we try

$$\begin{aligned} y_p &= Ae^{-x} + B\cos 3x + C\sin 3x, \\ y_p' &= -Ae^{-x} - 3B\sin 3x + 3C\cos 3x, \\ y_p'' &= Ae^{-x} - 9B\cos 3x - 9C\sin 3x. \end{aligned}$$

After we substitute these expressions into the differential equation in (6) and collect coefficients, we get

$$\begin{aligned} y_p'' - 3y_p' + 2y_p &= 6Ae^{-x} + (-7B - 9C)\cos 3x + (9B - 7C)\sin 3x \\ &= 3e^{-x} - 10\cos 3x. \end{aligned}$$

We equate the coefficients of the terms involving e^{-x}, those involving $\cos 3x$, and those involving $\sin 3x$. The result is the system

$$\begin{aligned} 6A &= 3, \\ -7B - 9C &= -10, \\ 9B - 7C &= 0 \end{aligned}$$

with solution $A = \frac{1}{2}$, $B = \frac{7}{13}$, and $C = \frac{9}{13}$. This gives the particular solution

$$y_p(x) = \tfrac{1}{2}e^{-x} + \tfrac{7}{13}\cos 3x + \tfrac{9}{13}\sin 3x,$$

which, however, does not have the required initial values in (6).

To satisfy those initial conditions, we begin with the *general* solution

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= c_1e^{x} + c_2e^{2x} + \tfrac{1}{2}e^{-x} + \tfrac{7}{13}\cos 3x + \tfrac{9}{13}\sin 3x, \end{aligned}$$

with derivative

$$y'(x) = c_1e^{x} + 2c_2e^{2x} - \tfrac{1}{2}e^{-x} - \tfrac{21}{13}\sin 3x + \tfrac{27}{13}\cos 3x.$$

The initial conditions in (6) lead to the equations

$$\begin{aligned} y(0) &= c_1 + c_2 + \tfrac{1}{2} + \tfrac{7}{13} = 1, \\ y'(0) &= c_1 + 2c_2 - \tfrac{1}{2} + \tfrac{27}{13} = 2 \end{aligned}$$

with solution $c_1 = -\frac{1}{2}$, $c_2 = \frac{6}{13}$. The desired particular solution is therefore

$$y(x) = -\tfrac{1}{2}e^{x} + \tfrac{6}{13}e^{2x} + \tfrac{1}{2}e^{-x} + \tfrac{7}{13}\cos 3x + \tfrac{9}{13}\sin 3x. \quad ■$$

Example 7 Find the general form of a particular solution of

$$y^{(3)} + 9y' = x\sin x + x^2e^{2x}. \tag{7}$$

Solution The characteristic equation $r^3 + 9r = 0$ has roots $r = 0$, $r = -3i$, and $r = 3i$. So the complementary function is

$$y_c(x) = c_1 + c_2 \cos 3x + c_3 \sin 3x.$$

The derivatives of the right-hand side in Eq. (7) involve the terms

$$\begin{gathered} \cos x, \quad \sin x, \quad x\cos x, \quad x \sin x, \\ e^{2x}, \quad xe^{2x}, \quad \text{and} \quad x^2e^{2x}. \end{gathered}$$

Because there is no duplication with the terms of the complementary function, the trial solution takes the form

$$y_p(x) = A\cos x + B\sin x + Cx\cos x + Dx\sin x + Ee^{2x} + Fxe^{2x} + Gx^2e^{2x}.$$

Upon substituting y_p in Eq. (7) and equating coefficients of like terms, we get seven equations determining the seven coefficients *A*, *B*, *C*, *D*, *E*, *F*, and *G*. ■

The Case of Duplication

Now we turn our attention to the situation in which Rule 1 does not apply: Some of the terms involved in $f(x)$ and its derivatives satisfy the associated homogeneous equation. For instance, suppose that we want to find a particular solution of the differential equation

$$(D - r)^3 y = (2x - 3)e^{rx}. \tag{8}$$

Proceeding as in Rule 1, our first guess would be

$$y_p(x) = Ae^{rx} + Bxe^{rx}. \tag{9}$$

This form of $y_p(x)$ will not be adequate because the complementary function of Eq. (8) is

$$y_c(x) = c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx}, \tag{10}$$

so substitution of (9) in the left-hand side of (8) would yield zero rather than $(2x - 3)e^{rx}$.

To see how to amend our first guess, we observe that

$$(D - r)^2[(2x - 3)e^{rx}] = [D^2(2x - 3)]e^{rx} = 0$$

by Eq. (13) of Section 2.3. If $y(x)$ is *any* solution of Eq. (8) and we apply the operator $(D - r)^2$ to both sides, we see that $y(x)$ is also a solution of the equation $(D - r)^5 y = 0$. The general solution of this *homogeneous* equation can be written as

$$y(x) = \underbrace{c_1e^{rx} + c_2xe^{rx} + c_3x^2e^{rx}}_{y_c} + \underbrace{Ax^3e^{rx} + Bx^4e^{rx}}_{y_p}.$$

Thus *every* solution of our original equation in (8) is the sum of a complementary function and a *particular solution* of the form

$$y_p(x) = Ax^3e^{rx} + Bx^4e^{rx}. \tag{11}$$

Note that the right-hand side in Eq. (11) can be obtained by multiplying each term of our first guess in (9) by the least positive integral power of x (in this case, x^3) that suffices to eliminate duplication between the terms of the resulting trial solution $y_p(x)$ and the complementary function $y_c(x)$ given in (10). This procedure succeeds in the general case.

To simplify the general statement of Rule 2, we observe that to find a particular solution of the nonhomogeneous linear differential equation

$$Ly = f_1(x) + f_2(x), \tag{12}$$

it suffices to find *separately* particular solutions $Y_1(x)$ and $Y_2(x)$ of the two equations

$$Ly = f_1(x) \quad \text{and} \quad Ly = f_2(x), \tag{13}$$

respectively. For linearity then gives

$$L[Y_1 + Y_2] = LY_1 + LY_2 = f_1(x) + f_2(x),$$

and therefore $y_p = Y_1 + Y_2$ is a particular solution of Eq. (12). (This is a type of "superposition principle" for nonhomogeneous linear equations.)

Now our problem is to find a particular solution of the equation $Ly = f(x)$, where $f(x)$ is a linear combination of products of the elementary functions listed in (4). Thus $f(x)$ can be written as a sum of terms each of the form

$$P_m(x)e^{rx}\cos kx \quad \text{or} \quad P_m(x)e^{rx}\sin kx, \tag{14}$$

where $P_m(x)$ is a polynomial in x of degree m. Note that any derivative of such a term is of the same form but with *both* sines and cosines appearing. The procedure by which we arrived earlier at the particular solution in (11) of Eq. (8) can be generalized to show that the following procedure is always successful.

RULE 2 Method of Undetermined Coefficients

If the function $f(x)$ is of either form in (14), take as the trial solution

$$\begin{aligned} y_p(x) = x^s[(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}\cos kx \\ + (B_0 + B_1x + B_2x^2 + \cdots + B_mx^m)e^{rx}\sin kx], \end{aligned} \tag{15}$$

where s is the smallest nonnegative integer such that such that no term in y_p duplicates a term in the complementary function y_c. Then determine the coefficients in Eq. (15) by substituting y_p into the nonhomogeneous equation.

In practice we seldom need to deal with a function $f(x)$ exhibiting the full generality in (14). The table in Fig. 2.5.1 lists the form of y_p in various common cases, corresponding to the possibilities $m = 0$, $r = 0$, and $k = 0$.

On the other hand, it is common to have

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x)$ and $f_2(x)$ are different functions of the sort listed in the table in Fig. 2.5.1. In this event we take as y_p the sum of the trial solutions for $f_1(x)$ and $f_2(x)$, choosing s *separately* for each part to eliminate duplication with the complementary function. This procedure is illustrated in Examples 8 through 10.

$f(x)$	y_p
$P_m = b_0 + b_1x + b_2x^2 + \cdots + b_mx^m$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)$
$a\cos kx + b\sin kx$	$x^s(A\cos kx + B\sin kx)$
$e^{rx}(a\cos kx + b\sin kx)$	$x^se^{rx}(A\cos kx + B\sin kx)$
$P_m(x)e^{rx}$	$x^s(A_0 + A_1x + A_2x^2 + \cdots + A_mx^m)e^{rx}$
$P_m(x)(a\cos kx + b\sin kx)$	$x^s[(A_0 + A_1x + \cdots + A_mx^m)\cos kx$ $+ (B_0 + B_1x + \cdots + B_mx^m)\sin kx]$

FIGURE 2.5.1. Substitutions in the method of undetermined coefficients.

Example 8 Find a particular solution of

$$y^{(3)} + y'' = 3e^x + 4x^2. \tag{16}$$

Solution The characteristic equation $r^3 + r^2 = 0$ has roots $r_1 = r_2 = 0$ and $r_3 = -1$, so the complementary function is

$$y_c(x) = c_1 + c_2x + c_3e^{-x}.$$

As a first step toward our particular solution, we form the sum

$$(Ae^x) + (B + Cx + Dx^2).$$

The part Ae^x corresponding to $3e^x$ does not duplicate any part of the complementary function, but the part $B + Cx + Dx^2$ must be multiplied by x^2 to eliminate duplication. Hence we take

$$\begin{aligned} y_p &= Ae^x + Bx^2 + Cx^3 + Dx^4, \\ y_p' &= Ae^x + 2Bx + 3Cx^2 + 4Dx^3, \\ y_p'' &= Ae^x + 2B + 6Cx + 12Dx^2, \quad \text{and} \\ y_p^{(3)} &= Ae^x + 6C + 24Dx. \end{aligned}$$

Substitution of these derivatives in Eq. (16) yields

$$2Ae^x + (2B + 6C) + (6C + 24D)x + 12Dx^2 = 3e^x + 4x^2.$$

The system of equations

$$\begin{aligned} 2A &= 3, & 2B + 6C &= 0, \\ 6C + 24D &= 0, & 12D &= 4 \end{aligned}$$

has the solution $A = \frac{3}{2}$, $B = 4$, $C = -\frac{4}{3}$, and $D = \frac{1}{3}$. Hence the desired particular solution is

$$y_p(x) = \tfrac{3}{2}e^x + 4x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{3}x^4.$$

■

Example 9 Determine the appropriate form for a particular solution of

$$y'' + 6y' + 13y = e^{-3x} \cos 2x.$$

Solution The characteristic equation $r^2 + 6r + 13 = 0$ has roots $-3 \pm 2i$, so the complementary function is

$$y_c(x) = e^{-3x}(c_1 \cos 2x + c_2 \sin 2x).$$

This is the same form as a first attempt $e^{-3x}(A \cos 2x + B \sin 2x)$ at a particular solution, so we must multiply by x to eliminate duplication. Hence we would take

$$y_p(x) = e^{-3x}(Ax \cos 2x + Bx \sin 2x).$$ ■

Example 10 Determine the appropriate form for a particular solution of the fifth-order equation

$$(D - 2)^3(D^2 + 9)y = x^2e^{2x} + x \sin 3x.$$

Solution The characteristic equation $(r - 2)^3(r^2 + 9) = 0$ has roots $r = 2, 2, 2, 3i$, and $-3i$, so the complementary function is

$$y_c(x) = c_1e^{2x} + c_2xe^{2x} + c_3x^2e^{2x} + c_4 \cos 3x + c_5 \sin 3x.$$

As a first step toward the form of a particular solution, we examine the sum

$$[(A + Bx + Cx^2)e^{2x}] + [(D + Ex) \cos 3x + (F + Gx) \sin 3x].$$

To eliminate duplication with terms of $y_c(x)$, the first part—corresponding to x^2e^{2x}—must be multiplied by x^3, and the second part—corresponding to $x \sin 3x$—must be multiplied by x. Hence we would take

$$y_p(x) = (Ax^3 + Bx^4 + Cx^5)e^{2x} + (Dx + Ex^2) \cos 3x + (Fx + Gx^2) \sin 3x.$$ ■

Variation of Parameters

Finally, let us point out the kind of situation in which the method of undetermined coefficients cannot be used. Consider, for example, the equation

$$y'' + y = \tan x, \tag{17}$$

which at first glance may appear similar to those considered in the preceding examples. Not so; the function $f(x) = \tan x$ has *infinitely many* linearly independent derivatives

$$\sec^2 x, \quad 2 \sec^2 x \tan x, \quad 4 \sec^2 x \tan^2 x + 2 \sec^4 x, \quad \ldots.$$

Therefore, we do not have available a *finite* linear combination to use as a trial solution.