Section 7.5 - Sum and Difference Formulas

$$\begin{cases} \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \end{cases}$$

Ex) Find $\cos(75^\circ)$

Solution:
$$\begin{aligned} \cos(75^\circ) &= \cos(45^\circ+30^\circ) \\ &= \cos(45^\circ)\cos(30^\circ) - \sin(45^\circ)\sin(30^\circ) \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{1}{2} \\ &= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} = \boxed{\frac{1}{4}(\sqrt{6}-\sqrt{2})} \end{aligned}$$

Ex) Find $\cos\left(\frac{\pi}{12}\right)$

Soln:
$$\begin{aligned} \cos\left(\frac{\pi}{12}\right) &= \cos\left(\frac{3\pi}{12}-\frac{2\pi}{12}\right) = \cos\left(\frac{\pi}{4}-\frac{\pi}{6}\right) \\ &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \boxed{\frac{1}{4}(\sqrt{6}+\sqrt{2})} \end{aligned}$$

Special:
$$\sin\left(\frac{\pi}{2}-\theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$$

$$\begin{cases} \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta) \\ \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \end{cases}$$

Ex) Find $\sin\left(\frac{7\pi}{12}\right)$

$$= \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right) = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$$

$$= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \boxed{\frac{1}{4}\left(\sqrt{2} + \sqrt{6}\right)}$$

Ex) Prove the identity $\frac{\cos(\alpha-\beta)}{\sin\alpha \sin\beta} = \cot\alpha \cot\beta + 1$

$$\Rightarrow \frac{\cos(\alpha-\beta)}{\sin\alpha \sin\beta} = \frac{\cos\alpha \cos\beta + \sin\alpha \sin\beta}{\sin\alpha \sin\beta}$$

$$= \frac{\cos\alpha}{\sin\alpha} \cdot \frac{\cos\beta}{\sin\beta} + \frac{\cancel{\sin\alpha}\cancel{\sin\beta}}{\cancel{\sin\alpha}\cancel{\sin\beta}}$$

$$= \boxed{\cot\alpha \cot\beta + 1}$$

Ex)

$$\tan(\alpha+\beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}$$

$$\tan(\alpha-\beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

Ex) Find $\sin\left(\cos^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{3}{5}\right)\right)$

Soln: Let $\alpha=\cos^{-1}\left(\frac{1}{2}\right)$ $\beta=\sin^{-1}\left(\frac{3}{5}\right)$

need angle in $[0,\pi]$ need angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$\Rightarrow$ Use $\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-\frac{1}{4}}=\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}$

$\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-\frac{9}{25}}=\sqrt{\frac{16}{25}}=\frac{4}{5}$

$\Rightarrow \sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$=\frac{\sqrt{3}}{2}\frac{4}{5}+\frac{1}{2}\cdot\frac{3}{5}=\boxed{\frac{4\sqrt{3}+3}{10}}$

Ex) Solve $\sin\theta+\cos\theta=1$

$\Rightarrow (\sin\theta+\cos\theta)^2=1$

$\Rightarrow \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=1$

$\Rightarrow 2\sin\theta\cos\theta+(\sin^2\theta+\cos^2\theta)=1$

$\Rightarrow 2\sin\theta\cos\theta+1=1$

$\Rightarrow 2\sin\theta\cos\theta=0$

$\Rightarrow \sin\theta=0$ or $\cos\theta=0$

$\Rightarrow \theta=0,\pi$ or $\theta=\frac{\pi}{2},\frac{3\pi}{2}$

Test $\Rightarrow$ Solutions are $\frac{\pi}{2}$ and 0.

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