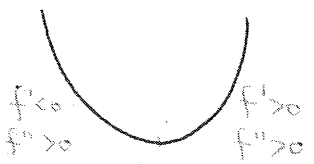
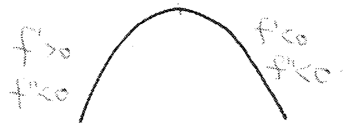
3.4. Concavity and Second Derivative.

Def. Let f be differentiable on I. • The graph of f is concave up on I if f' is increasing.

- The graph of f is concave down on I if f' is decreasing.
- If f' is constant the the graph of f is said to have no concavity.



concave up



concave down

Test for Concavity: Let f be twice differentiable on I. The graph of f is concave up if $f''>0$ on I, and is concave down if $f''<0$ on I.

Def. (Point of Inflection) A point of inflection is a point on the graph of f at which the concavity of f changes. If $(c, f(c))$ is a point of inflection on the graph of f, then either $f''=0$ or f'' is not defined at c.

Discussion

Second Derivative Test: Let c be critical value of f where $f(c)$ is def

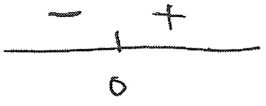
- If $f''(c)>0$, then f has a local min at $(c, f(c))$; 2. If $f''(c)>0$. the f has local

Find intervals of concave up / down, inflection points.

Ex. $f(x) = x^3 - 3x + 1$

$f'(x) = 3x^2 - 3$; $f''(x) = 6x$

$f''(x) = 0 \Rightarrow x = 0$.



f is concave up on $(0, \infty)$

f is concave down on $(-\infty, 0)$.

Inflection point : $x = 0$. or $(0, 1)$.

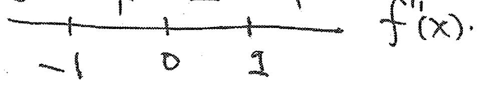
Ex. $f(x) = \dfrac{x}{x^2-1}$

$$f'(x) = \frac{x^2-1 - x\cdot 2x}{(x^2-1)^2} = \frac{-x^2-1}{(x^2-1)^2}.$$

$$f''(x) = \frac{-2x\cdot(x^2-1)^2 - (-x^2-1)\cdot 2(x^2-1)\cdot 2x}{(x^2-1)^4} = \frac{-2x(x^2-1) + (x^2+1)\cdot 4x}{(x^2-1)^3}$$

$$= \frac{-2x^3+2x+4x^3+4x}{(x^2-1)^3} = \frac{2x^3+6x}{(x^2-1)^3} = \frac{2x(x^2+3)}{(x^2-1)^3}$$

$f''(x) = 0 \Rightarrow x = 0$. $f''(x)$ is not defined when $x = \pm 1$.



$f''(-2) = \dfrac{-\cdot+}{+} < 0.$

$f''(-\frac{1}{2}) = \dfrac{(-)\cdot(+)}{(-)} > 0.$

$f''(\frac{1}{2}) = \dfrac{+\cdot+}{-} < 0$

$f''(2) = \dfrac{+\cdot+}{+} > 0.$

f concave up $(-1,0) \cup (1,\infty)$

concave down. $(-\infty,-1) \cup (0,1)$

Inflection point $(0, -1)$.

([illegible] $x = \pm 1$)

10 $f(x) = \frac{4x}{x^2+1}$.

$f'(x) = \frac{4(x^2+1) - 4x \cdot 2x}{(x^2+1)^2} = \frac{4x^2+4-8x^2}{(x^2+1)^2} = \frac{4-4x^2}{(x^2+1)^2}$

$f'(x) = 0 \Rightarrow x = \pm 1$.

$-$ $+$ $-$

-1 1 $f'(x)$

$\downarrow$ local min $\downarrow$ local max.

$f'(-2) < 0$

$f'(0) > 0$

$f'(2) < 0$

local min $f(-1) = \frac{-4}{2} = -2$ $(-1, -2)$

local max. $f(1) = \frac{4}{2} = 2$. $(1, 2)$.

f increases on $(-1, 1)$

f decreases on $(-\infty, -1) \cup (1, \infty)$.

$f''(x) = \frac{-8x \cdot (x^2+1)^2 - 4(1-x^2)\ 2(x^2+1) \cdot 2x}{(x^2+1)^4}$

$= \frac{-8x(x^2+1) - 16x(1-x^2)}{(x^2+1)^4} = \frac{-8x^3 - 8x - 16x + 16x^3}{(x^2+1)^4}$

$= \frac{8x^3 - 24x}{(x^2+1)^4} = \frac{8x(x^2-3)}{(x^2+1)^4} = 0$ $x = 0$. $x = \pm\sqrt{3}$

$-$ $+$ $-$ $+$

$-\sqrt{3}$ 0 $\sqrt{3}$ $f''(x)$

$f''(-2) = \frac{-+}{+} < 0$

$f''(-1) = \frac{(-)(-)}{+} > 0$

$f''(1) = \frac{+ \cdot -}{+} < 0$

$f''(2) > 0$

$f(-\sqrt{3}) = \frac{-4\sqrt{3}}{4} = -\sqrt{3}$

$f(0) = 0$

$f(\sqrt{3}) = \frac{4\sqrt{3}}{4} = \sqrt{3}$.

Concave up $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$

Concave down $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$.

Inflection points $(-\sqrt{3}, -\sqrt{3})$ $(0, 0)$. $(\sqrt{3}, \sqrt{3})$

Section 3.5 - Curve Sketching

Steps

① Find domain of $f(x)$

② Find Critical Values $\left(\text{Solve } f'(x)=0\right)$

③ Find inflection points $\left(\text{solve } f''(x)=0\right)$

④ Find vertical asymptotes, if any

⑤ Do limits

$\lim_{x \to \infty} f(x)$, $\lim_{x \to -\infty} f(x)$ if applicable

⑥ Create number line including information above

⑦ Sketch the graph

Example: Sketch

$$f(x) = \frac{x^2-x-2}{x^2-x-6} = \frac{(x-2)(x+1)}{(x-3)(x+2)}$$

① **Domain**: $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

② Critical points

$$f'(x) = \frac{-4(2x-1)}{(x^2-x-6)^2}$$

$$f'(x) = 0 \Rightarrow \boxed{x = \frac{1}{2}}$$

③ Inflection points

$$f''(x) = \frac{8(3x^2-3x+7)}{(x^2-x-6)^3}$$

$$f''(x) = 0 \Rightarrow 3x^2-3x+7 = 0$$

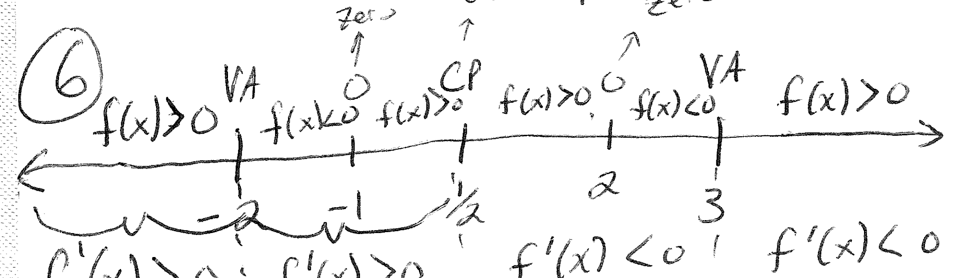
$$x = \frac{1}{2} \pm i\,\frac{5\sqrt{3}}{6}$$

$\Rightarrow$ No inflection points for $f'(c)=0$ but $x=-2$, $x=3$ inflection points

④ VA's at $x=-2$, $x=3$

⑤ $\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 1 \Rightarrow$ HA @ $y=1$

⑥



$f'(x)>0$ increasing $f''(x)>0$ concave up

$f'(x)>0$ increasing $f''(x)<0$ concave down

$f'(x)<0$ decreasing $f''(x)<0$ concave down

$f'(x)<0$ decreasing $f''(x)>0$ concave up

① ↳ inflection

① ↳ inflection

→ concavity changes!!