Sections 11.2, 11.3, 5, 4

11.2

Normalization Condition: $\int_0^1 r(x)\,\phi_n^2(x)\,dx = 1$ for $n \in \mathbb{N}$ ⊛

Example: $y'' + \lambda y = 0$ B.C.: $y(0) = 0$, $y(1) = 0$

We know that the solution is $y(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x$
and $\sqrt{\lambda} = n\pi$ from the B.C.'s $\Rightarrow \lambda_n = n^2\pi^2$
and so eigen functions are $k_n \sin(n\pi x) = \phi_n(x)$

So for normalization, use formula (recall general form $[p(x)y']' - q(x)y + \lambda r(x)y = 0$)

so for us $r(x) = 1$

$$\int_0^1 k_n^2 \sin^2 n\pi x\,dx$$

$$= k_n^2 \int_0^1 \left(\tfrac{1}{2} - \tfrac{1}{2}\cos(2n\pi x)\right)dx \quad \text{using trig identity}$$

$$= \tfrac{1}{2}k_n^2 \quad \Rightarrow \quad \tfrac{1}{2}k_n^2 = 1 \text{ by ⊛}$$

$$\Rightarrow k_n = \sqrt{2} \text{ for all } n$$

$$\Rightarrow \phi_n(x) = \sqrt{2}\sin(n\pi x) \quad \blacksquare$$

Non-Homogeneous Sturm-Liouville Problem

$$L[y] = -[p(x)y']' + q(x)y = \mu r(x)y + f(x) \qquad 0 \le f(x) \le 1, \ \mu \text{ given}$$

So solution is $y = \phi(x) = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\phi_n(x)$

with $c_n = \int_0^1 f(x)\,\phi_n(x)\,dx$

Section 5.4

Euler DE: $x^2y'' + \alpha x y' + \beta y = 0$

We want to use substitution $y = x^r,\ y' = rx^{r-1},\ y'' = r(r-1)x^{r-2}$

$$\Rightarrow x^2 \cdot r(r-1)x^{r-2} + \alpha x r x^{r-1} + \beta x^r = 0$$
$$\Rightarrow r(r-1)x^r + \alpha r x^r + \beta x^r = 0$$
$$\Rightarrow \left(r(r-1) + \alpha r + \beta\right)x^r = 0$$
$$\Rightarrow r(r-1) + \alpha r + \beta = 0$$
$$r^2 - r + \alpha r + \beta = 0$$
$$r^2 + (\alpha-1)r + \beta = 0$$
$$\Rightarrow r = \frac{-(\alpha-1)}{2} \pm \frac{\sqrt{(\alpha-1)^2 - 4\beta}}{2} \quad \star$$

Case 1: Real roots

$$y = C_1 x^{r_1} + C_2 x^{r_2} \qquad x > 0$$

Case 2: Equal Roots

$$y = (C_1 + C_2 \ln x)x^{r_1} \qquad x > 0$$

Case 3: Complex Roots ie) $r = \lambda \pm i\mu$

$$y = C_1 x^\lambda \cos(\mu \ln x) + C_2 x^\lambda \sin(\mu \ln x)$$

Example: $x^2y'' + xy' + y = 0$

* Try substitution to get answer.

Using $\star$, $r = \frac{-(1-1)}{2} \pm \frac{\sqrt{(1-1)^2 - 4}}{2}$

$\alpha = 1$

$\beta = 1$

$$= 0 \pm \frac{2i}{2} = \pm i \Rightarrow \text{complex roots}$$
$$\lambda = 0,\ \mu = 1$$

$$\Rightarrow y = C_1 \cos(\ln x) + C_2 \sin(\ln x) \qquad x > 0$$