PDE - HW#2

10.2) ~~18~~, 29
10.3) 4, ~~16~~, 17, 18
10.4) 2, 4, 6, 8, 12, 15, ~~16~~

18) $f(x) = \begin{cases} 0 & -2 \le x \le -1 \\ x & -1 < x < 1 \\ 0 & 1 \le x < 2 \end{cases}$ $\quad f(x+4) = f(x)$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} x\,dx = 0$$

$$a_n = \frac{1}{2}\int_{-1}^{1} x\cos\left(\frac{n\pi x}{L}\right)dx = 0$$

$$b_n = \frac{1}{2}\int_{-1}^{1} x\sin\left(\frac{n\pi x}{2}\right)dx = \frac{2}{n^2\pi^2}\left(2\sin\frac{n\pi}{2} - n\pi\cos\frac{n\pi}{2}\right)$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty}\left[\frac{4}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)$$

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19) $f(x) = \begin{cases} -1 & -2 \le x < 0 \\ 1 & 0 \le x < 2 \end{cases}$ $\quad f(x+4) = f(x)$

$$a_0 = \frac{1}{2}\int_{-2}^{0} -1\,dx + \frac{1}{2}\int_{0}^{2} 1\,dx = -\frac{x}{2}\Big|_{-2}^{0} + \frac{x}{2}\Big|_{0}^{2} = -1+1 = 0$$

$$a_n = \frac{1}{2}\int_{-2}^{0} -\cos\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2}\cos\left(\frac{n\pi x}{2}\right)dx = 0$$

$$b_n = \frac{1}{2}\int_{-2}^{0} -\sin\left(\frac{n\pi x}{2}\right)dx + \frac{1}{2}\int_{0}^{2}\sin\left(\frac{n\pi x}{2}\right)dx = 2\,\frac{1-\cos n\pi}{n\pi}$$

$$\Rightarrow f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\left(\frac{(n-1)\pi x}{2}\right)$$

29) a) Let $\vec{v}_1, \vec{v}_2, \vec{v}_3$ be mutually orthogonal vectors in 3 dimensions and $\vec{u}$ be any 3D vector.

$\Rightarrow$ $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are a basis for $\mathbb{R}^3$

$\Rightarrow$ from defn of basis, any $\vec{u}$ can be written as

$$\vec{u} = a_1 \vec{v}_1 + a_2 \vec{v}_2 + a_3 \vec{v}_3$$

Find a_i. Hint: Compute inner product $\vec{u} \cdot \vec{v}_i$

b) Eqn (10) is

$$\int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx = \frac{a_0}{2} \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) dx + \sum_{m=1}^{\infty} a_m \int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$
$$+ \sum_{m=1}^{\infty} b_m \int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx$$

Hint: Just Apply $(u, v) = \int_{-L}^{L} u(x) v(x)\, dx \rightarrow \circledast$ directly on $\underline{\qquad\qquad}$ parts above

c) (f, ψ_n) can be calculated just like in (b)

Hint: Use $\circledast$ above, with orthogonality conditions on p. 598 eqns 6, 7, 8 to calculate

$$a_n = \frac{(f, \phi_n)}{(\phi_n, \phi_n)} \qquad b_n = \frac{(f, \psi_n)}{(\psi_n, \psi_n)} \qquad a_0 = \frac{2\,(f, \phi_0)}{(\phi_0, \phi_0)}$$

17) Let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$

$$f(x)^2 = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\left[a_n^2\cos^2\left(\frac{n\pi x}{L}\right) + b_n^2 \sin^2\left(\frac{n\pi x}{L}\right)\right] + a_0\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$
$$+ \sum_{m,n=1;\, m\neq n} 2a_m a_n \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) + \sum_{\substack{m,n=1\\ m\neq n}} 2b_m b_n \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$
$$0 \quad + \sum_{m,n=1}^{\infty} 2a_m b_n \cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right) \quad 0$$

Integrating, orthogonality conditions

$$\int_{-L}^{L} f(x)^2\,dx = \int_{-L}^{L}\frac{a_0^2}{4}\,dx + \sum_{n=1}^{\infty}\left[\int_{-L}^{L} a_n^2\cos^2\left(\frac{n\pi x}{L}\right)dx + \int_{-L}^{L} b_n^2\sin^2\left(\frac{n\pi x}{L}\right)dx\right]$$
$$= \frac{a_0^2}{4}L + \sum_{n=1}^{\infty}\left[a_n^2 L + b_n^2 L\right]$$
$$\Rightarrow \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Lemma 4.15
Prop 7.6.4

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18) a) f, f' piecewise cont. on $-L \le x < L$, f periodic with period $2L$
Show that na_n and nb_n are bounded as $n \to \infty$

Proof: $f'(x) = \sum_{n=1}^{\infty} nb_n \cos(n\theta) - an a_n \sin(n\theta)$

$$a_n' = \frac{1}{L}\int_{-L}^{L} f'(t)\cos nt\,dt = \frac{1}{L}f(t)\cos nt\Big|_{-L}^{L} + \frac{1}{L}\int_{-L}^{L} f(t)n\sin nt\,dt = nb_n$$
$$b_n' = \frac{1}{L}\int_{-L}^{L} f'(t)\sin nt\,dt = \frac{1}{L}f(t)\sin nt\Big|_{-L}^{L} - \frac{1}{L}\int_{-L}^{L} f(t)n\cos nt\,dt = -na_n$$
$$a_0' = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt = 0$$

$$|a_n'| = |nb_n| \le \frac{1}{L}\int_{-L}^{L}|f'(t)\cos(nt)|\,dt \le \frac{1}{L}\int_{-L}^{L}|f'(t)|\,dt = 2\|f'\|_1 < \infty$$
$$|b_n'| = |na_n| \le \qquad = 2\|f'\|_1 < \infty$$

since piecewise cont. functions on finite intervals must be bounded

10.4 - 2, 4, 6, 8, 12, 15, 16

2) $x^3 - 2x + 1$

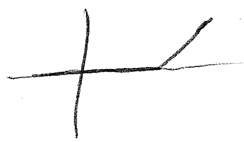
Function contains even/odd powers $\Rightarrow$ neither

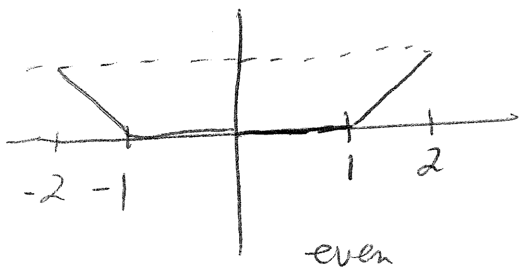
4) $\sec x = \frac{1}{\cos(x)}$ Quotient of two even functions is even

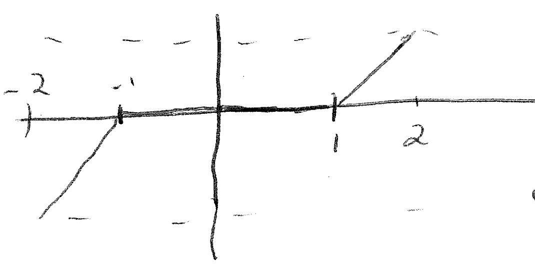
$\Rightarrow$ sec is even

6) e^{-x} Neither even/odd Taylor series

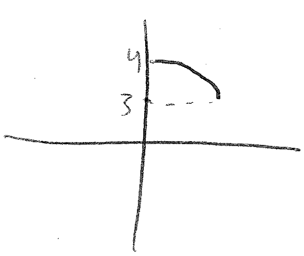
8) $f(x) = \begin{cases} 0 & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \end{cases}$

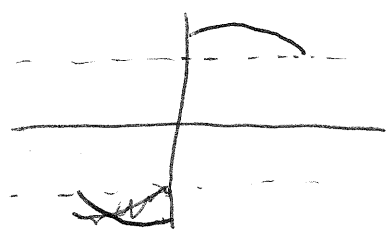






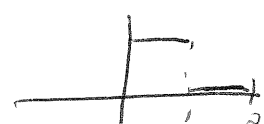
12) ~~4x~~ $f(x) = 4 - x^2 \quad 0 < x\ 1$





15) $f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$ cosine series, period 4

Sine coeffs $= 0$



$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{2}{2}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ (-1)^{k+1} & n = 2k-1 \quad k = 1, 2, \ldots \end{cases}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx = 1\int_0^1 1\,dx = 1$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{(2n-1)\pi}\cos\left(\frac{(2n-1)\pi x}{2}\right)$$