PDE - HW 3

10.4) ~~19~~, ~~31~~, ~~35~~, 36, ~~37~~
10.5) ~~1~~, ~~3~~, 7, ~~10~~, ~~11~~, ~~22~~, ~~23~~
10.6) ~~2~~, ~~3~~, ~~4~~

19) $f(x) = \begin{cases} 0 & 0 < x < \pi \\ 1 & \pi < x < 2\pi \\ 2 & 2\pi < x < 3\pi \end{cases}$ period, 6π

a) $L = 3\pi$, sines $\Rightarrow a_n = 0$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{3\pi}\int_\pi^{2\pi}\sin\left(\frac{nx}{3}\right)dx + \frac{2}{3\pi}\int_{2\pi}^{3\pi} 2\sin\left(\frac{nx}{3}\right)dx$$

$$= -\frac{2}{3\pi}\cdot\frac{3}{n}\cos\left(\frac{nx}{3}\right)\Big|_\pi^{2\pi} - \frac{4}{3\pi}\cdot\frac{3}{n}\cos\left(\frac{nx}{3}\right)\Big|_{2\pi}^{3\pi}$$

$$= -\frac{2}{n\pi}\cos\left(\frac{2\pi n}{3}\right) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{3}\right) - \frac{4}{n\pi}\cos\left(\frac{3\pi n}{3}\right) + \frac{4}{n\pi}\cos\left(\frac{2\pi n}{3}\right)$$

$$= \frac{2}{n\pi}\cos\left(\frac{2\pi n}{3}\right) + \frac{2}{n\pi}\cos\left(\frac{n\pi}{3}\right) - \frac{4}{n\pi}\cos(n\pi)$$

$$\Rightarrow f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[\cos\left(\frac{n\pi}{3}\right) + \cos\left(\frac{2\pi n}{3}\right) - 2\cos(n\pi)\right]\sin\left(\frac{nx}{3}\right)$$

b) picture

31) $\int_{-L}^{L} f(x)dx = \int_{-L}^{0} f(x)dx + \int_0^L f(x)dx \stackrel{?}{=} 0$

$\Rightarrow$ we can show $\int_{-L}^{0} f(x)dx = -\int_0^L f(x)dx$

Let $u = -x$, $du = -1\,dx$

$$LHS = \int_{-L}^{0} f(x)dx = -\int_L^0 f(-u)du$$
$$= \int_0^L f(-u)du$$
$$= \int_0^L f(-x)\,dx \quad \text{rename}$$
$$= -\int_0^L f(x)\,dx \quad \text{odd func}$$

$\boxed{\text{QED}}$

35) From the Fourier series of square wave in Ex 1, 10.3 show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

Proof: $f(x) = \begin{cases} 0 & -L < x < 0 \\ L & 0 < x < L \end{cases} \xrightarrow{L = \pi/2} \begin{cases} 0 & -\frac{\pi}{2} < x < 0 \\ \pi/2 & 0 < x < \pi/2 \end{cases}$

$$f(x) = \frac{L}{2} + \frac{2L}{\pi}\left(\sin\left(\frac{\pi x}{L}\right) + \frac{1}{3}\sin\left(\frac{3\pi x}{L}\right) + \frac{1}{5}\sin\left(\frac{5\pi x}{L}\right) + \cdots\right)$$

$$f\left(\frac{L}{2}\right) = \frac{L}{2} + \frac{2L}{\pi}\left(\sin\left(\frac{\pi}{2}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$L = \frac{\pi}{2} = f\left(\frac{L}{2}\right) = f(\pi/4) = \pi/2$$

$$\Rightarrow \frac{\pi}{2} = \frac{\pi}{4} + \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

36) $f(x) = \begin{cases} -x & -2 \le x < 0 \\ x & 0 \le x < 2 \end{cases}$

$L = 2$

$$\Rightarrow f(x) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi x}{2}$$

At $x = 0$

$$\Rightarrow 0 = f(0) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\Rightarrow \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

37) a) $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad 0 \le x \le L$

$$f(x)^2 = \left(\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)\right)^2$$

$$= \left(b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + b_3 \sin\left(\frac{3\pi x}{L}\right) + \dots\right)\left(b_1 \sin\left(\frac{\pi x}{L}\right) + b_2 \sin\left(\frac{2\pi x}{L}\right) + b_3 \sin\left(\frac{3\pi x}{L}\right) + \dots\right)$$

$$= b_1^2 \sin^2\left(\frac{\pi x}{L}\right) + b_1 b_2 \sin\left(\frac{\pi x}{L}\right) + b_3 b_2 + b_2^2 \sin^2\left(\frac{2\pi x}{L}\right) + \dots \quad (*)$$

$$\Rightarrow \frac{2}{L}\int_0^L f(x)^2\,dx = \frac{2}{L}\sum_{n=1}^{\infty}\int_0^L (*)\,dx$$

Using orthogonality, $\int_0^L \sin \sin = \begin{cases} 0 & m \neq n \\ \frac{L}{2} & n = m \end{cases} \quad \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx = \frac{L}{2}$

$\Rightarrow$ cross terms $\sin \sin = 0$

$$\Rightarrow \frac{2}{L}\int_0^L f(x)^2\,dx = \frac{2}{L}\cdot\frac{L}{2}\sum_{n=1}^{\infty} b_n^2 = \sum_{n=1}^{\infty} b_n^2$$

result same for cosine except $= \sum_{n=1}^{\infty} a_n^2$

b) Eq 9 is $f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right) \qquad f(x) = \begin{cases} x & -L < x < L \\ 0 & f(L) = f(-L) = 0 \end{cases}$

From (a), $\frac{2}{L}\int_0^L f(x)^2\,dx = \sum_{n=1}^{\infty} b_n^2$

$b_n^2 = \frac{4L^2}{n^2\pi^2} \quad \Rightarrow \quad \frac{2}{L}\int_0^L x^2\,dx = \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$

$$\Rightarrow \int_0^L x^2\,dx = \frac{2L^3}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \left.\frac{x^3}{3}\right|_0^L = \text{//}$$

$$\frac{L^3}{3} = \text{//}$$

$$\Rightarrow \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2} \quad \blacksquare$$

10.5

1) $x u_{xx} + u_t = 0$ Consider soln $u(x,t) = h(t)\phi(x)$

$\Rightarrow x h \phi'' + \phi h' = 0$

$\Rightarrow \frac{1}{h\phi}\left[x h \phi'' + \phi h'\right] = 0$

$\Rightarrow x\frac{\phi''}{\phi} + \frac{h'}{h} = 0 \Rightarrow \frac{x\phi''}{\phi} = -\frac{h'}{h} = \lambda$

$\Rightarrow x\phi'' - \lambda\phi = 0$ and $h' + \lambda h = 0$

3) $u_{xx} + u_{xt} + u_t = 0$ $u(x,t) = h(t)\phi(x)$

$\Rightarrow h\phi'' + h'\phi' + h'\phi = 0$

$\Rightarrow h\phi'' + h'(\phi' + \phi) = 0$

$\Rightarrow \frac{h\phi''}{\phi' + \phi} + h' = 0 \Rightarrow \frac{\phi''}{\phi' + \phi} + \frac{h'}{h} = 0$

$\Rightarrow \frac{\phi''}{\phi' + \phi} = -\frac{h'}{h} = \lambda \Rightarrow \phi'' - \lambda(\phi' + \phi) = 0$
$h' + \lambda h = 0$

7) $100 u_{xx} = u_t$ $0 < x < 1$

$u(0,t) = 0$ $u(1,t) = 0$ $t > 0$

$u(x,0) = \sin(2\pi x) - \sin(5\pi x)$ $0 \le x \le 1$

Consider $u(x,t) = h(t)\phi(x)$

$\Rightarrow 100 h \phi'' = h'\phi$

$\Rightarrow \frac{\phi''}{\phi} = \frac{1}{100}\frac{h'}{h} = -\lambda \Rightarrow$ $\phi'' + \lambda\phi = 0$
$h' + 100\lambda h = 0$

$\phi'' + \lambda\phi = 0$

$\Rightarrow \phi(x) = c_1\cos\sqrt{\lambda}x + c_2\sin\sqrt{\lambda}x$

$\rightarrow 0 = c_1$

$\rightarrow 0 = c_2\sin\sqrt{\lambda}$

$\Rightarrow \sqrt{\lambda} = n\pi$

$\Rightarrow \lambda = n^2\pi^2$

$\Rightarrow \phi(x) = c_2\sin(n\pi x)$

$h' + 100\lambda h = 0$

$\Rightarrow \frac{h'}{h} = -\lambda 100$

$\frac{1}{h}\,dh = -\lambda\,dt \cdot 100$

$\ln h = -\lambda t \cdot 100$

$\Rightarrow h(t) = Ce^{-\lambda 100 t}$

$\Rightarrow u_n(x,t) = c_n e^{-n^2\pi^2 100 t}\sin(n\pi x)$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 100 t}\sin(n\pi x)$

IC $\Rightarrow \sum_{n=1}^{\infty} c_n \sin(n\pi x) = \sin(2\pi x) - \sin(5\pi x)$

$\Rightarrow c_2 = -c_5 = 1$ all other $c_n = 0$

$\Rightarrow u(x,t) = e^{-400\pi^2 t}\sin(2\pi x) - e^{-2500\pi^2 t}\sin(5\pi x)$

---

10) rod $40cm = L$, $u(0,t) = u(40,t) = 0$, $\alpha^2 = 1$

$u(x,0) = \begin{cases} x & 0 \le x < 20 \\ 40 - x & 20 \le x < 40 \end{cases}$

$u(x,t) = h(t)\phi(x)$

$\frac{h'}{h} = -\lambda$

$\Rightarrow$

$\phi(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$h(t) = Ce^{-\lambda t}$

$\Rightarrow c_1 = 0$

$\phi(x) = c_2\sin(\sqrt{\lambda}40)$

$\Rightarrow 40\sqrt{\lambda} = n\pi$

$\Rightarrow \lambda = \frac{n^2\pi^2}{1600}$

$\Rightarrow u_n(x,t) = c_n e^{-\lambda t}\sin\left(\frac{n\pi x}{40}\right)$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2 t}{1600}}\sin\left(\frac{n\pi x}{40}\right)$

$$c_n = \frac{2}{40}\int_0^{20} x \sin\left(\frac{n\pi x}{40}\right) dx + \frac{2}{40}\int_{20}^{40} (40-x)\sin\left(\frac{n\pi x}{40}\right) dx$$

$$= \cancel{\frac{-40}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{80}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \cancel{\frac{40}{n\pi}\cos\left(\frac{n\pi}{2}\right)} + \frac{80}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) - \underbrace{\cancel{\frac{80}{n^2\pi^2}\sin(n\pi)}}_{0}$$

$$c_n = \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) e^{\frac{-\pi^2 n^2 t}{1600}} \sin\left(\frac{n\pi x}{40}\right)}$$

$$u(x,0) = \sum_{n=1}^{\infty} \frac{160}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{40}\right)$$

11) $u(x,0) = \begin{cases} 0 & 0 \le x < 10 \\ 50 & 10 \le x < 30 \\ 0 & 30 < x \le 40 \end{cases}$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n e^{\frac{-n^2\pi^2 t}{1600}} \sin\left(\frac{n\pi x}{40}\right)$

$\Rightarrow c_n = \frac{5}{2}\int_{10}^{30} \sin\left(\frac{n\pi x}{40}\right) dx = 100\left(\frac{\cos(n\pi/4) - \cos(\frac{3n\pi}{4})}{n\pi}\right)$

$\Rightarrow u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty} \frac{\cos(n\pi/4) - \cos(\frac{3n\pi}{4})}{n} e^{\frac{-n^2\pi^2 t}{1600}} \sin\left(\frac{n\pi x}{40}\right)$

---

22) $\alpha^2(u_{xx}+u_{yy}) = u_t \Rightarrow \alpha^2 \nabla^2 u = u_t$

$u(x,y,t) = \phi(x,y)h(t)$

$\Rightarrow \frac{\nabla^2\phi}{\phi} = \frac{1}{\alpha^2}\frac{h'}{h} = -\lambda \Rightarrow \boxed{h' + \alpha^2\lambda h = 0}$

$\phi(x,y) = \psi(x)\xi(y)$

$\Rightarrow \nabla^2\phi + \lambda\phi = 0$

$\Rightarrow \phi_{xx} + \phi_{yy} + \lambda\phi = 0$

$\Rightarrow \psi''\xi + \psi\xi'' + \lambda\psi\xi = 0$

$\frac{\psi''}{\psi} + \frac{\xi''}{\xi} + \lambda = 0 \Rightarrow \frac{\psi''}{\psi} = -\frac{\xi''}{\xi} + \lambda = -\mu$

$\Rightarrow \psi'' + \mu\psi = 0$ and $\xi'' + (\mu + \lambda)\xi = 0$

23) $\alpha^2\left[u_{rr} + \left(\frac{1}{r}\right)u_r + \left(\frac{1}{r^2}\right)u_{\theta\theta}\right] = u_t$

$$u(r,\theta,t) = R(r)\Theta(\theta)T(t)$$

Sub $\Rightarrow$ $\alpha^2\left[R''\Theta T + \frac{1}{r}R'\Theta T + \frac{1}{r^2}R\Theta''T\right] = R\Theta T'$

Divide by $R\Theta T$

$$\Rightarrow \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta^2}{\Theta} = \frac{T'}{\alpha^2 T} = -\lambda^2$$

It follows that $T' + \alpha^2\lambda^2 T = 0$

and $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\lambda^2$

$$\Rightarrow r^2\frac{R''}{R} + r\frac{R'}{R} + \lambda^2 r^2 = -\frac{\Theta''}{\Theta} = \mu^2$$

$$\Rightarrow r^2R'' + rR' + (\lambda^2r^2 - \mu^2)R = 0$$

$$\Theta'' + \mu^2\Theta = 0$$

$$T' + \alpha^2\lambda^2 T = 0$$

10.6

2) $\alpha^2 u_{xx} = u_t \quad u(0,t) = 30\ ,\ u(40,t) = -20$

p 637-8 (P634 top!!)

Steady state solution is a constant/linear

$\Rightarrow \quad u''(x,t) = 0\ ,\ u''(x) = 0$ no time dep. $t \to \infty$

$$\frac{d^2u}{dx^2} = 0 \Rightarrow \text{on } 0 < x < 40$$

$u(0) = 30$
$u(40) = -20$

$\hookrightarrow \quad u(x) = c_1 x + c_2$

$\Rightarrow \quad c_2 = 30\ ,\quad u(x) = c_1 x + 30$

$\Rightarrow \quad -20 = 40c_1 + 30 \Rightarrow \frac{-50}{40} = c_1 \Rightarrow c_1 = -\frac{5}{4}$

$\Rightarrow \boxed{u(x) = -\frac{5}{4}x + 30}$

3) $u(x) = c_1 x + c_2$ from $u''(x) = 0$

B.C.) $u_x(0,t) = 0\ ,\ u(L,t) = 0$

$\hookrightarrow c_1 = 0 \Rightarrow u(x) = c_2$

BC2 $\Rightarrow \quad 0 = c_2 \Rightarrow u(x) \equiv 0$, is steady state

4) $u(x) = c_1 x + c_2$

$u_x(0,t) = 0 \Rightarrow c_1 = 0$

$u(L,t) = T$

$u(x) = c_2 \overset{BC2}{\Rightarrow} T = c_2$

$\Rightarrow u(x) = T$ is steady state