10.6) 9a, 11a, 13a,b, 15, 21
10.7) 2a, 4a, 6a, 8a, 9, 13, 15, 16, 17

10.6

9a) $L=20,\ T_1=0,\ T_2=60,\ \alpha^2=.86$ (Aluminum)
p 624 chart.

Steady-State

$V''(x)=0 \Rightarrow$ ~~$v(x)=c_1x+c_2x$~~

p634 Eqn(11) $\Rightarrow v(x)=(T_2-T_1)\frac{x}{L}+T_1$

$\Rightarrow V(x)=(60-0)\frac{x}{20}+0=3x$

So $u(x,t)=V(x)+w(x,t)$

So $\alpha^2 w_{xx}=w_t$ with B.C. $w(0,t)=w(20,t)=0$
IC. $w(x,0)=u(x,0)-v(x)=25-3x$

Sep. of vars

$\phi(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x \qquad h(t)=Ce^{-\frac{.86n^2\pi^2}{400}t}$

$\phi(x)=c_2\sin\left(\frac{n\pi x}{20}\right)$

$$\Rightarrow w(x,t)=\sum_{n=1}^{\infty}c_n e^{-\frac{.86n^2\pi^2}{400}t}\sin\left(\frac{n\pi x}{20}\right)$$

$$c_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx=\frac{2}{20}\int_0^{20}(25-3x)\sin\left(\frac{n\pi x}{20}\right)dx$$

$$=\frac{10}{n\pi}(5-7(-1)^n)$$

$$\Rightarrow u(x,t)=3x+\sum_{n=1}^{\infty}\frac{10}{n\pi}(5+7(-1)^n)e^{-\frac{.86n^2\pi^2}{400}t}\sin\left(\frac{n\pi x}{20}\right)$$

11a) $L = 30$, $\alpha^2 = 1$ $u(0,t) = 30$, $u(30,t) = 0$

$$u(x,0) = \frac{x(60-x)}{30}$$

$$u(x,t) = v(x) + w(x,t)$$

$$V(x) = (T_2 - T_1)\frac{x}{L} + T_1 = (-30)\frac{x}{30} + 30 = 30 - x$$

$$\alpha^2 w_{xx} = w_t$$

$$w(x,t) = \sum_{n=1}^{\infty} c_n e^{\frac{-n^2\pi^2 t}{900}} \sin\left(\frac{n\pi x}{30}\right)$$

$$w(x,0) = u(x,0) - V(x) = \frac{x(60-x)}{30} - 30 + x$$

$$= \frac{60x - x^2 - 900 + 30x}{30}$$

$$= \frac{-x^2 + 90x - 900}{30}$$

$$c_n = \frac{2}{30}\int_0^{30} w(x,0)\sin\left(\frac{n\pi x}{30}\right)dx$$

$$= 60\,\frac{2(1-\cos(n\pi)) - n^2\pi^2\cos(n\pi)}{n^3\pi^3}$$

$$\Longrightarrow u(x,t) = 30 - x + \frac{60}{\pi^3}\sum_{n=1}^{\infty} \frac{2(1-\cos(n\pi)) - n^2\pi^2\cos(n\pi)}{n^3\pi^3} e^{\frac{-n^2\pi^2 t}{900}} \sin\left(\frac{n\pi x}{30}\right)$$

12a) length L, $u(x,0) = \sin\left(\frac{\pi x}{L}\right) \quad 0 \le x \le L$

$$u_x(0,t) = 0 \qquad \alpha^2 u_{xx} = u_t$$
$$u_x(L,t) = 0$$

$$\Rightarrow \phi(x) = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$$
$$\phi'(x) = -c_1(\sin\sqrt{\lambda}x)\sqrt{\lambda} + c_2\sqrt{\lambda}\cos\sqrt{\lambda}\,x$$

BC.1 $\Rightarrow c_2 = 0$

BC 2 $\Rightarrow \sqrt{\lambda}\cancel{L} = (n\cancel{-\tfrac{1}{2}})\pi \Rightarrow \sqrt{\lambda} = \frac{(n\cancel{-\tfrac{1}{2}})\pi}{L}$

$$h(t) = C e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \qquad \phi(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=0}^{\infty} c_n e^{\frac{-n^2\pi^2\alpha^2}{L^2}t} \cos\left(\frac{n\pi x}{L}\right)$$

$$c_n = \frac{2}{L}\int_0^L \sin\left(\frac{\cancel{n}\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2(1+\cos(n\pi))}{\pi - n^2\pi} \qquad n=0 \Rightarrow \frac{2}{\pi}$$

$$\Rightarrow u(x,t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} c_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t}\cos\left(\frac{n\pi x}{L}\right)$$

$$c_n = \begin{cases} 0 & n \text{ odd} \\ -\dfrac{4}{(n^2-1)\pi} & n \text{ even} \end{cases}$$

b) Steady state, $t \to \infty$

$$\Rightarrow u(x) = \frac{2}{\pi}$$

15) L, IC: $f(x) =$ for $0 \le x \le L$

BC 1 $\Rightarrow u(0,t) = 0$ $\qquad \alpha^2 u_{xx} = u_t$

BC 2 $\Rightarrow u_x(L,t) = 0$

Assume $u(x,t) = \phi(x)\, h(t)$

$\Rightarrow \alpha^2 \phi''(x)\, h(t) = \phi(x)\, h(t)$

$\Rightarrow \dfrac{\phi''(x)}{\phi(x)} = \dfrac{h'(t)}{h(t)} \dfrac{1}{\alpha^2} = -\lambda$

$\phi(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

BC 1 $\Rightarrow c_1 = 0$

$\Rightarrow \phi(x) = c_2 \sin\sqrt{\lambda}\, x$

$\phi'(x) = c_2\sqrt{\lambda} \cos\sqrt{\lambda}\, x$

BC2 $\Rightarrow 0 = c_2\sqrt{\lambda} \cos\sqrt{\lambda} L$

$\Rightarrow \sqrt{\lambda} L = (n - \tfrac{1}{2})\pi \Rightarrow \sqrt{\lambda} = \dfrac{(n-\frac{1}{2})\pi}{L}$

$\phi(x) = c_2 \sin\left(\dfrac{(n-\frac{1}{2})\pi x}{L}\right)$

$h(t) = C e^{-\frac{(n-\frac{1}{2})^2 \pi^2 \alpha^2}{L^2} t}$

$\Rightarrow u_n(x,t) = e^{-\frac{(n-\frac{1}{2})^2 \pi^2 \alpha^2}{L^2} t} \sin\left(\dfrac{(n-\frac{1}{2})\pi x}{2}\right)$

b) $c_n = \dfrac{2}{L} \displaystyle\int_0^L f(x) \sin\left(\dfrac{(n-\frac{1}{2})\pi x}{L}\right) dx$

21) $u_t = \alpha^2 u_{xx} + s(x)$

$$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \end{cases} \qquad u(x,0) = f(x)$$

$u(x,t) = V(x) + w(x,t)$

<u>PDE</u>: $u_t - \alpha^2 u_{xx} = s(x)$

Steady state $t \to \infty$, so $\frac{d}{dt} u = 0$

$\Rightarrow -\alpha^2 u_{xx} = s(x)$

oneVar. $\Rightarrow \alpha^2 V''(x) + s(x) = 0$ is ODE

with $V(0) = T_1$, $V(L) = T_2$ by ⊛

Transient solves Heat eqn

$\Rightarrow \alpha^2 w_{xx} = w_t$ and $\left.\begin{array}{l} w(0,t) = 0 \\ w(L,t) = 0 \end{array}\right\}$ P634 eqn 14,15

$w(x,0) = f(x) - v(x)$

10.7

2a) $L=10, a=1$

$$f(x) = \begin{cases} \frac{4x}{L} & 0 \le x \le L/4 \\ 1 & L/4 < x < \frac{3L}{4} \\ \frac{4(L-x)}{L} & \frac{3L}{4} \le x \le L \end{cases} \qquad \left.\begin{array}{l} u(0,t)=0 \\ u(10,t)=0 \\ u_t(x,0)=0=g(x) \end{array}\right\} \text{I.C.}$$

$$a^2 u_{xx} = u_{tt} \Rightarrow u_{xx} = u_{tt}$$

$$\phi''(x) h(t) = \phi(x) h''(t)$$

$$\Rightarrow \frac{\phi''(x)}{\phi(x)} = \frac{h''(t)}{h(x)} = -\lambda \Rightarrow \begin{array}{l} \phi''(x) + \lambda \phi(x) = 0 \\ h''(t) + \lambda h(t) = 0 \end{array}$$

Eigenvalue problem

$$\Rightarrow \phi(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x \qquad \lambda = \frac{n^2\pi^2}{L^2}$$

$$BC's \Rightarrow \phi(x) = c_2 \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow h''(t) + \frac{n^2\pi^2}{L^2} h(t) = 0$$

$$\Rightarrow h(t) = c_1 \cos\left(\frac{n\pi t}{L}\right) + c_2 \sin\left(\frac{n\pi t}{L}\right)$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi t}{L}\right)\right)$$

I.C. satisfied if $\quad f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi}{L} \sin\left(\frac{n\pi x}{L}\right) = 0$$

$$\Rightarrow c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\left(\int_0^{L/4} \frac{4x}{L}\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/4}^{3L/4} \sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{3L}{4}}^{L} \frac{4L-4x}{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= 8\,\frac{\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)}{n^2\pi^2}$$

$$\Rightarrow u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)}{n^2} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi t}{L}\right)$$

4a) $f(x) = u(x,0) = \begin{cases} 1 & \frac{L}{2} - 1 < x < \frac{L}{2} + 1 \\ 0 & \text{otherwise} \end{cases}$

$u(0,t) = 0$
$u(L,t) = 0$ $\quad u_t(x,0) = 0$

Since B.C.'s same, u_t same,

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a t}{L}\right)$$

$$f(x) = u(x,0) \Rightarrow c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_{\frac{L}{2}-1}^{\frac{L}{2}+1} \sin\left(\frac{n\pi x}{L}\right) dx = 4\,\frac{\sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}\right)}{n\pi}$$

$$\Rightarrow u(x,t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[\sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a t}{L}\right)$$

6a) $f(x) = 0$, $g(x) = (2a)$ $\quad u(0,t) = u(L,t) = 0$

$u_t(x,0) = g$

Solve eigen value problem

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(A_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi a t}{L}\right) + B_n \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi a t}{L}\right) \right)$$

$$\Rightarrow \quad 0 = f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi a}{2} \sin\left(\frac{n\pi \cancel{x}}{L}\right)$$

$$\Rightarrow \quad c_n \frac{n\pi a}{L} = \underbrace{\frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right)}\, dx$$

$$c_n \frac{n\pi a}{L} = \frac{8}{\pi^2} \frac{\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)}{n^2}$$

$$\Rightarrow u(x,t) \frac{8L}{a\pi^3} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{4}\right) + \sin\left(\frac{3n\pi}{4}\right)}{n^3} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi a t}{L}\right)$$

8a) $f(x)=0,\ g(x)=(4a)$

$u_t(x,0)=$

$\Rightarrow$ Soln same since B.C.'s same

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a t}{L}\right)\frac{n\pi a}{L}$$

$$c_n\frac{n\pi a}{L}=\underbrace{\frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}_{\text{same as }(4a)}$$

$$\Rightarrow c_n\frac{\pi n a}{L}=\frac{4\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}\right)}{n\pi} \Rightarrow c_n=\frac{4L}{n^2\pi^2 a}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}\right)$$

$$\Rightarrow u(x,t)=\frac{4L}{\pi^2 a}\sum_{n=1}^{\infty}\left[\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi}{L}\right)\right]\cdot\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi a t}{L}\right)$$

---

9) $u(0,t)=0$ $\quad u_t(x,0)=0$ $\quad \sqrt{\lambda_n}=\frac{(n-\frac{1}{2})L}{\pi}$, sin in x variable (see 2a)

$u_x(L,t)=0$ $\quad u(x,0)=f(x)$

$a^2u_{xx}=u_{tt}$

$$\Rightarrow u(x,t)=\sum_{n=1}^{\infty}\left(A_n\sin(\sqrt{\lambda_n}x)\cos(\sqrt{\lambda_n}at)\right)+B_n\sin(\sqrt{\lambda_n}x)\sin(\sqrt{\lambda_n}at)$$

Using I.C.'s

$$f(x)=u(x,0)=\sum_{n=1}^{\infty}A_n\sin(\sqrt{\lambda_n}x)$$

$$\cancel{g(x)=u_t(x,0)=0}=\sum_{n=1}^{\infty}\sqrt{\lambda_n}aB_n\sin(\sqrt{\lambda_n}x)$$

$$\Rightarrow c_n=\frac{2}{L}\int_0^L f(x)\sin\sqrt{\lambda_n}x\,dx$$

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13) $\alpha^2u_{xx}=u_t$ $\quad$ C.O.V $\Rightarrow \xi=x-at,\ \eta=x+at$

a) $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$ $\qquad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t}+\frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t}$ (Chain Rule)

$\Rightarrow u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi+u_\eta$ $\qquad u_t=u_\xi\xi_t+u_\eta\eta_t=-au_\xi+au_\eta$

$u_{xx}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$ $\qquad u_{tt}=a^2(u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta})$

$\Rightarrow$ PDE $\Rightarrow 4a^2u_{\xi\eta}=0 \Rightarrow u_{\xi\eta}=0$

$\square$

b) Integrate both sides w.r.t. η

$\Rightarrow \int u_{\xi\eta}\, d\eta = \int 0\, d\eta$

$\Rightarrow u_\xi = \omega(\xi)$ arb. funct.

Integ wrt ξ

$\Rightarrow u(\xi,\eta) = \int \omega(\xi)\, d\xi + \psi(\eta)$

$\Rightarrow u(\xi,\eta) = \phi(\xi) + \psi(\eta)$

$\Rightarrow u(x,t) = \phi(x-at) + \psi(x+at)$

15) Steel wire, $L=5$, $T=50^{lb}$, $\gamma = .026\ ^{lb}/_{ft}$

$a^2 = \frac{T}{\rho}$ $\quad \rho = \frac{\gamma}{g}$ $\quad g = 32.2\ ft/s^2$

a) $a^2 = \frac{Tg}{\gamma} \Rightarrow a = \sqrt{\frac{Tg}{\gamma}} = 248\ ft/s$

b) $\omega_n = \frac{n\pi a}{L} = 49.8\pi n\ rad/s$

c) $\Rightarrow a = \sqrt{\frac{g(T+\Delta T)}{\gamma}}$ so affects ω_n

Modes are proportional to $M_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

no affect since a not involved

16) $\alpha^2 u_{xx} = u_{tt}$ $\quad u(x,0) = f(x)$

$u_t(x,0) = 0$

a) $u(x,t) = \phi(x-at) + \psi(x+at)$

IC 1 $\Rightarrow u(x,0) = \phi(x) + \psi(x) = f(x)$

$u_t(x,t) = -a\phi'(x-at) + a\psi'(x+at)$

$u_t(x,0) = -a\phi'(x) + a\psi'(x) = a(-\phi'(x) + \psi'(x)) = 0$

$\Rightarrow -\phi'(x) + \psi'(x) = 0$

b) $\phi(x) + \psi(x) = f(x)$

$-\phi'(x) + \psi'(x) = 0$

Integration

$-\phi(x) + \psi(x) = \int_0^x 0\, dx + C \longrightarrow \begin{array}{l} \phi(x) + \psi(x) = f(x) \\ -\phi(x) + \psi(x) = \tilde{C} \end{array}$

$\Rightarrow \psi(x) = \tilde{C} + \phi(x)$

$\psi(x) = \frac{1}{2} f(x) + \frac{1}{2}\tilde{C}$ and $\phi(x) = \frac{1}{2} f(x) - \frac{1}{2}\tilde{C}$

$\Rightarrow u(x,t) = \phi(x-at) + \psi(x+at)$

$= \frac{1}{2} f(x-at) \mp \frac{1}{2}\tilde{C} + \frac{1}{2} f(x+at) + \frac{1}{2}\tilde{C}$

$= \frac{1}{2}\left[f(x-at) + f(x+at)\right]$

c) $f(x) = \begin{cases} 2 & -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow f(x-at) = \begin{cases} 2 & -1 < x - at < 1 \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow f(x-at) = \begin{cases} 2 & -1 + at < x < 1 + at \\ 0 & \text{otherwise} \end{cases}$

$f(x+at)$ Same way.

d) In book

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17) $a^2 u_{xx} = u_{tt}$ $\quad u(x,0) = 0$, $\quad u_t(x,0) = g(x)$

$u(x,t) = \phi(x-at) + \psi(x+at)$

IC 1) $\Rightarrow \boxed{\phi(x) + \psi(x) = 0}$

IC 2) $\Rightarrow -a\phi'(x-at) + a\psi'(x+at) = u_t(x,t)$

$\Rightarrow \boxed{-a\phi'(x) + a\psi'(x) = g(x)}$

b) $\phi(x)+\psi(x)=0$

$$\Rightarrow \phi'(x)+\psi'(x)=0 \Rightarrow \psi'(x)=-\phi'(x) \quad \bigstar$$

$\Rightarrow$ using $-a\phi'(x)+a\psi'(x)=g(x)$

$$\xrightarrow{\text{by } \bigstar} \quad -2a\phi'(x)=g(x)$$

$$\Rightarrow \quad \phi'(x)=-\frac{1}{2a}g(x)$$

by integrating $\phi(x)=-\dfrac{1}{2a}\displaystyle\int_{x_0}^{x} g(\xi)\,d\xi+\phi(x_0)$

$$\phi(x)+\psi(x)=0$$

$$\Rightarrow -\phi(x)=\psi(x) \Rightarrow \psi(x)=\frac{1}{2a}\int_{x_0}^{x} g(\xi)\,d\xi-\phi(x_0)$$

c) $u(x,t)=\phi(x-at)+\psi(x+at)$

$$=-\frac{1}{2a}\int_{x_0-at}^{x-at} g(\xi)\,d\xi+\phi(x_0+at)+\frac{1}{2a}\int_{x_0+at}^{x+at} g(\xi)\,d\xi-\phi(x_0+at)$$

$$=\frac{1}{2a}\int_{x-at}^{x_0-at} g(\xi)\,d\xi+\frac{1}{2a}\int_{x_0+at}^{x+at} g(\xi)\,d\xi$$

x_0 arbitrary

$$=\frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi$$