10.8) ~~2, 5, 7~~, 10a

11.1) ~~8, 9, 10~~, 12, ~~13, 14, 15, 17, 18, 20~~

2) $\nabla^2 u = 0$

$u_{xx} + u_{yy} = 0$

$u(0,y) = 0 \qquad u(x,0) = h(x)$

$u(a,y) = 0 \qquad u(x,b) = 0$

$\phi''(x) + \lambda\phi(x) = 0 \implies \lambda = \left(\frac{n\pi}{a}\right)^2,\ \phi(x) = c_2 \sin\left(\frac{n\pi x}{a}\right)$

$\psi''(y) - \lambda\psi(y) = 0 \implies \psi(y) = \sinh\left(\frac{n\pi}{a}(b-y)\right)$

(*) $\psi(y) = a_1 \cosh\left(\frac{n\pi}{a}(y-L)\right) + a_2 \sinh\left(\frac{n\pi}{a}(y-L)\right)$ are linearly independent translation for convenience (PDE is translation invariant)

$$\implies u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi}{a}(b-y)\right)$$

$$\text{B.C.} \implies h(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\frac{n\pi b}{a}$$

$$\implies c_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a h(x)\sin\left(\frac{n\pi x}{a}\right)dx$$

5) $u_{rr} + \left(\frac{1}{r}\right)u_r + \left(\frac{1}{r^2}\right)u_{\theta\theta} = 0$, $\boxed{\text{outside}}$ $r \geq a \qquad 0 \leq \theta < 2\pi$

$u(a,\theta) = f(\theta)$

$u(r,\theta) = \phi(\theta)G(r)$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dG}{dr}\right)\phi(\theta) + \frac{1}{r^2}G\frac{d^2\phi}{d\theta^2} = 0 \qquad \left(\text{divide by } \frac{1}{r^2}G\phi\right)$$

$$\implies \frac{r}{G}\frac{d}{dr}\left(r\frac{dG}{dr}\right) = -\frac{1}{\phi}\frac{d^2\phi}{d\theta^2} = \lambda = (n^2)$$

$$\implies r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - n^2 G = 0 \qquad \text{Euler DE.}$$

Sub $G = r^p \implies [p(p-1) + p - n^2]r^p = 0 \implies p = \pm n$

Assuming $n \neq 0$ we have $G = c_1 r^n + c_2 r^{-n}$ (*)

$\phi = d_1 \cos n\theta + d_2 \sin(n\theta)$

$\longrightarrow$

From (A) we throw out r^n since $r^n \to \infty$ as $r \to \infty$

$$\Rightarrow G = c_2 r^{-n}$$

$$\Rightarrow u(r,\theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty} r^{-n}(c_n \cos(n\theta) + k_n \sin(n\theta))$$

$$\Rightarrow \frac{1}{a^n} c_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$$

$$\frac{1}{a^n} k_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$$

$\blacksquare$

7) $u(r,\theta)$, $0<r<a$, $0<\theta<\alpha$

$$\left.\begin{aligned} u(r,0) &= 0 \\ u(r,\alpha) &= 0 \end{aligned}\right\} 0 \le r < a$$

$$u(a,\theta) = f(\theta) \} \; 0 \le \theta \le \alpha$$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$u(r,\theta) = G(r)\phi(\theta)$$

$$\Rightarrow \frac{r^2G'' + rG'}{G} = -\frac{\phi''}{\phi} = \lambda$$

$$\Rightarrow \phi'' + \lambda\phi = 0 \Rightarrow \phi(\theta) = c_1\cos\sqrt{\lambda}\,\theta + c_2\sin\sqrt{\lambda}\,\theta$$

$$\Rightarrow \phi(\theta) = c_2\sin\left(\frac{n\pi\theta}{\alpha}\right) \quad \sqrt{\lambda} = \frac{n\pi}{\alpha}$$

$$G = d_1 r^{\sqrt{\lambda}} + d_2 r^{-\sqrt{\lambda}} \longrightarrow \infty \text{ for } r \to 0$$

$$\Rightarrow G = d_1 r^{\frac{n\pi}{\alpha}}$$

$$\Rightarrow u(r,\theta) = \sum_{n=1}^{\infty} c_n r^{\frac{n\pi}{\alpha}} \sin\left(\frac{n\pi\theta}{\alpha}\right)$$

and $$\frac{1}{a^{\frac{n\pi}{\alpha}}} c_n = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)\sin\left(\frac{n\pi\theta}{\alpha}\right)d\theta$$

$\blacksquare$

10a) $u(x,y)$, $\nabla^2 u = 0$, $\begin{matrix} 0 < x < a \\ 0 < y < b \end{matrix}$

$u_x(0,y) = 0 \quad u_x(a,y) = f(y) \qquad 0 < y < b$

$u_y(x,0) = 0 \quad u_y(x,b) = 0 \qquad 0 \le x \le a$

$u_{xx} + u_{yy} = 0$

$$\Rightarrow \frac{\phi''(x)}{\phi(x)} = -\frac{\psi''(y)}{\psi(y)} = \lambda$$

$$\Rightarrow \phi''(x) - \lambda \phi(x) = 0 \Rightarrow \phi(x) = c_1 \cosh(\sqrt{\lambda} x) + c_2 \sinh(\sqrt{\lambda} x)$$

$$\psi''(y) + \lambda \psi(y) = 0 \Rightarrow \psi(y) = d_1 \cos(\sqrt{\lambda} y) + d_2 \sin(\sqrt{\lambda} y)$$

+ const solution

y BC $\Rightarrow \psi(y) = d_1 \cos\left(\frac{n\pi y}{b}\right)$

x BC $\Rightarrow \phi(x) = c_1 \cosh\left(\frac{n\pi x}{b}\right)$

$$\Rightarrow \boxed{u(x,y) = \text{cst} \sum_{n=1}^{\infty} c_n \cosh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right)}$$

$$\Rightarrow u_x(x,y) = \sum_{n=1}^{\infty} c_n \frac{n\pi}{b} \sinh\left(\frac{n\pi x}{b}\right) \cos\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow \boxed{\frac{c_n \cdot n\pi}{b} \sinh\left(\frac{n\pi a}{b}\right) = \frac{2}{b} \int_0^b f(y) \cos\left(\frac{n\pi y}{b}\right) dy}$$

11.1

8) $y'' + \lambda y = 0 \qquad y'(0) = 0,\ y(1) + y'(1) = 0$

$\lambda = -\mu^2$

a) 1) $y(x) = c_1 \cosh\sqrt{-\lambda}\, x + c_2 \sinh\sqrt{-\lambda}\, x$

$BC \Rightarrow c_2 = 0$

$BC \Rightarrow c_1(\cosh\mu + \mu\sinh\mu) + c_2(\sinh\mu + \mu\cosh\mu) = 0$

0

$\Rightarrow c_1(1 + \mu\tanh(\mu)) = 0$

$\mu\tanh\mu \geq 0 \Rightarrow c_1 = 0$ No nontrivi. solns

2) $y(x) = c_1\cos\sqrt{\lambda}\, x + c_2\sin\sqrt{\lambda}\, x$

$BC \Rightarrow c_2 = 0$

$BC \Rightarrow c_1(\cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda}) + c_2(\sin\sqrt{\lambda} + \sqrt{\lambda}\cos\sqrt{\lambda}) = 0$

0

$\Longrightarrow \cos\sqrt{\lambda} - \sqrt{\lambda}\sin\sqrt{\lambda} = 0$

$\Rightarrow 1 - \sqrt{\lambda}\tan\sqrt{\lambda} = 0 \qquad \left(\mu_n \approx (n-1)\pi = \sqrt{\lambda}\right)$ for large n

Solve $\sqrt{\lambda}\tan\sqrt{\lambda} = 1$

b) 3) $\lambda = 0 \Rightarrow c_1 = c_2 = 0$

c) $\cos(\sqrt{\lambda}\, x)$

d) from (a) $\lambda_n \approx (n-1)^2\pi^2$

9) $y'' + \lambda y = 0 \qquad y(0) - y'(0) = 0$
$\qquad\qquad\qquad\qquad y(1) + y'(1) = 0$

a) 1 $y(x) = c_1 \cosh \mu x + c_2 \sinh \mu x$
$y'(x) = c_1 \mu \sinh \mu x + c_2 \mu \cosh \mu x$

$BC \Rightarrow \quad 0 = c_1 \mp c_2 \qquad\qquad \mu = \sqrt{\lambda}$

$\Rightarrow c_1 \cosh \mu + c_2 \sinh \mu + c_1 \mu \sinh \mu + c_2 \mu \cosh \mu$

$\Rightarrow c_1 (\cosh \mu + \sinh \mu + \mu \sinh \mu + \mu \cosh \mu) = 0$

$\Rightarrow \quad 1 + \tanh \mu + \mu \tanh \mu + \mu = 0 \qquad \mu = -1$

$\Rightarrow 1 + \mu + \tanh(\mu)(1+\mu) = (1+\mu)(1+\tanh \mu)$

2 $y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$
$y'(x) = -c_1 \sqrt{\lambda} \sin \sqrt{\lambda} x + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$

$BC \Rightarrow 0 = c_1 - c_2 \sqrt{\lambda}$

$BC \Rightarrow c_1 \cos \sqrt{\lambda} + c_2 \sin \sqrt{\lambda} - c_1 \sqrt{\lambda} \sin \sqrt{\lambda} + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} = 0$

$\Rightarrow c_2 \sqrt{\lambda} \cos \sqrt{\lambda} + c_2 \sin \sqrt{\lambda} - c_2 \lambda + c_2 \sqrt{\lambda} \cos \sqrt{\lambda} = 0$

$\Rightarrow c_2 \left(2 \sqrt{\lambda} \cos \sqrt{\lambda} + \sin \sqrt{\lambda} - \lambda \sin \sqrt{\lambda} \right) = 0$
(*)

$\Rightarrow \phi_n(x) = \sin \sqrt{\lambda_n} x + \sqrt{\lambda_n} \cos \sqrt{\lambda_n} x$

where $\sqrt{\lambda_n}$ satisfies (*)

b) $X(x) = c_1 x + c_2 \quad \Rightarrow \quad 0 = c_2 \mp c_1 \quad \Rightarrow c_1 = c_2$
$\qquad\qquad\qquad\qquad\qquad 0 = c_1 + c_2 + c_1 \Rightarrow 2c_1 = -c_2 \Rightarrow 3c_1 = 0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow c_1 = 0 \Rightarrow c_2 = 0$

c) (Book)

d) $\lambda_n = (n-1)^2 \pi^2$

10) $y'' - \lambda y = 0$ $\qquad y(0) + y'(0) = 0$, $y(1) = 0$

Similar to others (8, 9) except "$-$" sign in front of λ switches cos/sin to cosh/sinh and cosh/sinh to cos/sin

Same approach

(4)

$P(x)y'' + Q(x)y' + R(x)y = 0$

12) $y'' - 2xy' + \lambda y = 0$

$\Rightarrow P = 1,\ Q = -2x,\ R = \lambda$

$\Rightarrow \mu' = -2x\mu$

$\mu(x) = \frac{1}{P(x)} e^{\int_{x_0}^{x} \frac{Q(s)}{P(s)} ds} \Rightarrow \mu(x) = ce^{-x^2}$

$\Rightarrow [\mu P(x) y']' + \mu(x) R(x) y = 0$ (*)

$\Rightarrow [e^{-x^2} y']' + \lambda e^{-x^2} y = 0$

$\mu = \frac{1}{x^2} e^{\int_{x_0}^{x} \frac{x}{x^2} dx}$

13) $x^2y'' + xy' + (x^2 - v^2)y = 0$

$\Rightarrow P = x^2,\ Q = \cancel{\quad} x,\ R = (x^2 - v^2)$ $\boxed{\mu = \frac{1}{x^2} e^{\ln x} = \frac{1}{x}}$

$\Rightarrow \cancel{x^2\mu' = (x - x^2)\mu}$

$\cancel{\frac{\mu'}{\mu} = \frac{1}{x} - 1} \cancel{\Rightarrow \ln|\mu| = \ln x - x} \cancel{\Rightarrow e^{\ln x - x} = e^{\ln x}e^{-x} = xe^{-x}}$

$\cancel{\Rightarrow \ln\mu = xe^{-x}}$

$\Rightarrow \left[\frac{x^2}{x} y'\right]' + \frac{1}{x}(x^2 - v^2) y = 0$

14) $xy'' + (1-x)y' + \lambda y = 0$

$P(x) = x,\ Q(x) = 1 - x,\ R(x) = \lambda$

$\Rightarrow x\mu'(x) = -x\mu(x)$

$\Rightarrow \mu(x) = ce^{-x}$

(*) $\Rightarrow [xe^{-x} y']' + \lambda e^{-x} y = 0$

15) $(1-x^2)y'' - xy' + \alpha^2 y = 0$

$P(x) = 1-x^2$, $Q(x) = -x$, $R(x) = \alpha^2$

$\Rightarrow (1-x^2)\mu'(x) = (-x+2x) = x\mu(x)$

$\Rightarrow \int \frac{1}{\mu}\, d\mu = \int \frac{x}{1-x^2}\, dx$

$\Rightarrow \mu(x) = \frac{C}{\sqrt{|1-x^2|}}$

Chebyshev defined for $|x| \le 1$

$\Rightarrow \left[\sqrt{1-x^2}\, y'\right]' + \frac{\alpha^2}{\sqrt{1-x^2}} y = 0$

17) $y'' - 2y' + (1+\lambda)y = 0$, $y(0) = 0$, $y(1) = 0$

Let $y = s(x)u \Rightarrow y' = s'u + su'$

$y'' = s''u + 2s'u' + su''$

ODE $\Rightarrow s''u + 2s'u' + su'' - 2(s'u + su') + (1+\lambda)su = 0$

$\Rightarrow su'' + \underbrace{(2s' - 2s)}_{\circledast} u' + [s'' - 2s' + (1+\lambda)s]u = 0$

$\circledast$ disappears if $s' = s \Rightarrow s = e^x$

with $s(x) = e^x$

$e^x u'' + [e^x - 2e^x (1+\lambda)e^x]u = 0$

$\Rightarrow e^x[u'' + [1 - 2 + 1 + \lambda]u] = 0$

$\Rightarrow e^x[u'' + \lambda u] = 0$

$\Rightarrow u'' + \lambda u = 0$

$\Rightarrow u'' + \lambda u = 0 \quad u(0) = u(1) = 0$

$\Rightarrow$ we have sin, $\sqrt{\lambda} = \frac{n\pi}{L} = n\pi$

Original prob.

$\Rightarrow 1 + \lambda = 1 + n^2\pi^2$, $\phi(x) = c_2 e^x \sin(n\pi x)$

c) 2^{nd} order Homog ODE const. coeffs

$$\Rightarrow r^2 - 2r + (1+\lambda) = 0$$

$$\Rightarrow r = 1 \pm \sqrt{-\lambda}$$

1. $\lambda = 0 \Rightarrow y = c_1 e^x + c_2 x e^x$ from ODE

$BC \Rightarrow c_1 = c_2 = 0$

2. $\lambda < 0$

$$y = c_1 e^{1+\sqrt{-\lambda}\,x} + c_2 e^{1-\sqrt{-\lambda}\,x}$$

$BC \Rightarrow c_1 = c_2 = 0$

3. $\lambda > 0$

$$y = c_1 e^x \cos\sqrt{\lambda}\,x + c_2 e^x \sin\sqrt{\lambda}\,x$$

$BC \Rightarrow c_1 = 0$

$c_2 e \sin\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = n\pi$

19) $y'' + y' + \lambda(y' + y) = 0 \quad y'(0) = 0 \quad y(1) = 0$

$$\Rightarrow y'' + (1+\lambda)y' + \lambda y = 0$$

$$\Rightarrow r^2 + (1+\lambda)r + \lambda = 0$$

$r_1 = 1, \; r_2 = -\lambda$

For $\lambda \neq 1 \Rightarrow y(x) = c_1 e^{-x} + c_2 e^{-\lambda x}$

$BC \Rightarrow c_1 + c_2 = 0, \; c_1 e^{-1} + c_2 e^{-\lambda} = 0$

$\Rightarrow e^{-1} = e^{-\lambda} \Rightarrow \lambda = 1$ contradiction

$\lambda = 1 \Rightarrow y(x) = c_1 e^{-x} + c_2 x e^{-x}$

$BC \Rightarrow c_1 = 0, \quad c_2 e^{-1} = 0 \Rightarrow c_2 = 0$ no solns.

20) $x^2y'' - \lambda(xy'-y) = 0 \qquad y(1)=0,\ y(2)-y'(2)=0$

$$x^2y'' - \lambda x y' + \lambda y = 0$$

$$P(x) = x^2,\ Q(x) = -\lambda x,\ R(x) = \lambda$$

$$\cancel{\mu(x) = \left(\frac{1}{x^2}\right) e^{\int_{x_0}^{x}\left(-\frac{\lambda}{s}\right)ds}}$$

$$x^2\mu'(x) = (-\lambda x - x^2)\,\mu$$

$$\frac{1}{\mu}\,d\mu = \frac{-\lambda x - x^2}{x^2}\,dx$$

$$\frac{1}{\mu}\,d\mu = -\frac{\lambda}{x} - 1\ dx$$

$$\ln|\mu| = -x - \lambda\ln|x| + C$$

$$\Rightarrow |\mu| = Ce^{-x}e^{-\lambda\ln|x|} = Ce^{-x}|x|^{-\lambda}$$

Soln

$$= C_1X + C_2X^{\lambda}$$

$$\lambda = 0 \Rightarrow y = C_1x + C_2,\ y' = C_1$$

$$\Rightarrow c_1 + c_2 = 0$$

$$2c_1 + c_2 - c_1 = 0 \qquad \Rightarrow c_1 =$$

$$\Rightarrow c_1 + c_2 = 0$$

$$\phi(x) = x - 1$$