$\boxed{HW\#6}$

11.2) $\cancel{1}, \cancel{3}, \cancel{4}, \cancel{8}, \cancel{7}, \cancel{9}, \cancel{10}, \cancel{12}, \cancel{20}, 23$

11.3) $\cancel{2}, \cancel{3}, \cancel{7}, \cancel{11}, \cancel{19}, \cancel{20}, \cancel{22}$

11.2

1) $y'' + \lambda y = 0 \qquad y(0) = 0, \; y'(1) = 0$

Usual eigenvalue problem $\Rightarrow$ $y(x) = c_1 \cos\sqrt{\lambda}x + \sin\sqrt{\lambda}x$

B.C.'s $\Rightarrow$ Sin eigenfct. and $\sqrt{\lambda} = (n - \frac{1}{2})\pi$ b/c

$\Rightarrow \phi_n(x) = c_n \sin((n-\frac{1}{2})\pi x)$

Orthogonal/Normalization $\Rightarrow \int_0^1 r(x)\,\phi_n^2(x)\,dx = 1$ ⊛

where $[p(x)y']' - q(x)y + \lambda r(x) y = 0$

So for us, $r(x) = 1 \Rightarrow \int_0^1 \phi_n^2(x)\,dx = \int_0^1 c_n^2 \sin^2((n-\frac{1}{2})\pi x)\,dx$

$= c_n^2 \int_0^1 \left(\frac{1}{2} - \frac{1}{2}\cos(2(n-\frac{1}{2})\pi x)\right) dx = c_n^2 \cdot \frac{1}{2} \overset{⊛}{=} 1$

$\Rightarrow c_n = \sqrt{2} \Rightarrow \phi_n(x) = \sqrt{2}\sin((n-\frac{1}{2})\pi x)$

3) $y'' + \lambda y = 0 \qquad y'(0) = 0, \; y'(1) = 0$

Usual eigenvalue problem, from B.C.'s $\Rightarrow$ cos eigfncts, $\sqrt{\lambda} = n\pi$

$\Rightarrow \phi_n(x) = c_n \cos(n\pi x)$

Note: $\lambda = 0$ also has soln $\Rightarrow \phi_0(x) = c$ c a const.

but normalization requires $c = 1 \Rightarrow \phi_0(x) = 1$

We have $r(x) = 1$ again, so by ⊛,

$c_n^2 \int_0^1 \cos^2(n\pi x)\,dx = c_n^2 \cdot \frac{1}{2} \overset{⊛}{=} 1$

$\Rightarrow c_n = \sqrt{2}$

$\Rightarrow \phi_n(x) = \sqrt{2}\cos(n\pi x)$

4) Determine normalized eigenfunction

$y'' + \lambda y = 0 \qquad y'(0) = 0, \quad y'(1) + y(1) = 0$

From 11.1.8, $\phi_n(x) = k_n \cos\sqrt{\lambda_n}\, x$

with λ_n satisfying $\cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\sin\sqrt{\lambda_n} = 0$ ⊕

By ⊛ in (1), $k_n^2 \int_0^1 \cos\sqrt{\lambda_n}\, x \, dx = k_n^2 \dfrac{\cos\sqrt{\lambda_n}\sin\sqrt{\lambda_n} + \sqrt{\lambda_n}}{2\sqrt{\lambda_n}}$

⊛ $\sin^2 u = \frac{1}{2} - \frac{1}{2}\cos 2u$

By ⊕ $\Rightarrow \cos\sqrt{\lambda_n} = \sqrt{\lambda_n}\sin\sqrt{\lambda_n}$

$= k_n^2 \dfrac{1 + \sin^2\sqrt{\lambda_n}}{2} \overset{⊛}{=} k_n^2 \dfrac{3 - \cos 2\sqrt{\lambda_n}}{4} \overset{⊛}{=} 1$

$\Rightarrow k_n = \dfrac{2}{\sqrt{3 - \cos 2\sqrt{\lambda_n}}} = \dfrac{\sqrt{2}}{\sqrt{1 + \sin^2\sqrt{\lambda_n}}}$

$\Rightarrow \phi_n(x) = \dfrac{\sqrt{2}\cos\sqrt{\lambda_n}\, x}{\sqrt{1 + \sin^2\sqrt{\lambda_n}}}$

5) $y'' - 2y' + (1+\lambda)y = y'' - 2y' + y + \lambda y = 0$ (ODE)

From problem 17, $\phi_n(x) = k_n e^x \sin(n\pi x)$

(ODE) $\Rightarrow e^{-2x} y'' - 2e^{-2x} y' + e^{-2x} y + e^{-2x}\lambda y = 0$

$\Rightarrow [e^{-2x} y']' + e^{-2x} y + e^{-2x}\lambda y = 0 \Rightarrow r(x) = e^{-2x}$

By ~~orthog.~~ normal condition: $\int_0^1 r(x)\, \phi_n^2(x)\, dx = k_n^2 \int_0^1 e^{-2x} e^{2x} \sin^2(n\pi x)\, dx = \frac{1}{2}k_n^2 = 1$

$\Rightarrow k_n^2 = 2 \Rightarrow k_n = \sqrt{2}$

$\Rightarrow \phi_n(x) = \sqrt{2}\, e^x \sin(n\pi x)$

6) $f(x) = 1, \quad 0 \le x \le 1$

$\phi_n(x) = \sqrt{2}\sin((n-\tfrac{1}{2})\pi x) \quad n \in \mathbb{N}, \quad r(x) = 1$

From Eq.(34)

$$C_m = \int_0^1 f(x)\,\phi_m(x)\,dx = \int_0^1 \sqrt{2}\sin\left((n-\tfrac{1}{2})\pi x\right)dx = \frac{2\sqrt{2}}{(2m-1)\pi}$$

$$\Rightarrow a_n = \frac{2\sqrt{2}}{(2n-1)\pi}$$

$$\Rightarrow 1 = \frac{2\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin((n-\tfrac{1}{2})\pi x)$$

7) $f(x) = x, \quad 0 \le x \le 1$

$\phi_n(x) = \sin((n-\tfrac{1}{2})\pi x)\sqrt{2}$

$$\Rightarrow C_m = \int_0^1 x\sqrt{2}\sin\left((n-\tfrac{1}{2})\pi x\right)dx = \frac{4\sqrt{2}(-1)^{n-1}}{(2n-1)^2\pi^2}$$

$$\Rightarrow f(x) = \frac{4\sqrt{2}.\sqrt{2}}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin((n-\tfrac{1}{2})\pi x)\,dx$$

9) $f(x) = \begin{cases} 2x & 0 \le x < \tfrac{1}{2} \\ 1 & \tfrac{1}{2} \le x \le 1 \end{cases}$

$$C_m = \sqrt{2}\int_0^{1/2} 2x\sin((n-\tfrac{1}{2})\pi x) + \sqrt{2}\int_{1/2}^1 \sin((n-\tfrac{1}{2})\pi x)$$

$$= \frac{8}{(2m-1)^2\pi^2}\left[\sin\left(\frac{m\pi}{2}\right) - \cos\left(\frac{m\pi}{2}\right)\right]$$

10) $f(x) = 1 \quad 0 \le x \le 1 \quad \phi_n(x) = \dfrac{\sqrt{2}\cos\sqrt{\lambda_n}\,x}{\sqrt{1+\sin^2\sqrt{\lambda_n}}}$ (Prob #4)

$\Rightarrow f(x) = \sum_{n=1}^{\infty} c_n\phi_n(x)$, $\quad c_n = \int_0^1 f(x)\,\phi_n(x)\,dx = \int_0^1 K_n\cos\sqrt{\lambda_n}\,x\,dx$

$\Rightarrow C_n = K_n\dfrac{\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}}$

$\Rightarrow C_n = \dfrac{\sqrt{2}\sin\sqrt{\lambda_n}}{\sqrt{1+\sin^2\sqrt{\lambda_n}}\,\sqrt{\lambda_n}}$ and $f(x) = \sum_{n=1}^{\infty} C_n\phi_n$

12) $f(x) = 1-x \quad 0 \le x \le 1 \qquad \phi_n(x) = \frac{\sqrt{2}\cos\sqrt{\lambda_n}\,x}{\sqrt{1+\sin^2\sqrt{\lambda_n}}}$

$$c_n = \int_0^1 f(x)\phi_n(x)\,dx = \int_0^1 (1-x) k_n \cos\sqrt{\lambda_n}\,x\,dx$$

$$\Rightarrow \quad c_n = k_n \frac{1-\cos\sqrt{\lambda_n}}{(\sqrt{\lambda_n})^2} = \frac{\sqrt{2}\,(1-\cos\sqrt{\lambda_n})}{\lambda_n\sqrt{1+\sin^2\sqrt{\lambda_n}}}$$

$$\Rightarrow \quad f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

for the calculated c_n

20) a) Suppose λ is not simple, then $\exists\ \phi_1, \phi_2$ (Thm 11.2.3)

b) ~~Suppose~~ that are linearly independent with eigenvalue λ

Wronskian $\Rightarrow \phi_1(x)\phi_2'(x) - \phi_2(x)\phi_1'(x) = W(\phi_1,\phi_2)(x)$ ⊕

Each ϕ_i satisfies $a_1 y(0) + a_2 y'(0) = 0$

$$\Rightarrow \left.\begin{array}{l} a_1\phi_1(0) + a_2\phi_1'(0) = 0 \\ a_1\phi_2(0) + a_2\phi_2'(0) = 0 \end{array}\right\} \circledast$$

from ⊛, if a_1 or a_2 is zero, $\Rightarrow$ ⊕ $= 0$

So if $a_2 \ne 0 \Rightarrow W(\phi_1,\phi_2)(0) = -\frac{a_1}{a_2}\phi_1(0)\phi_2(0) + \frac{a_1}{a_2}\phi_1(0)\phi_2(0)$
(use ⊛ to get)
$= 0$

c) By Thm in book $W(\phi_1,\phi_2) \equiv 0 \quad \forall x \in [0,1]$

$\Rightarrow$ linear dependence, contradiction to linear indep

$\Rightarrow \lambda$ is simple

23) a) Let λ be an eigenvalue, $\phi(x) = U(x) + iV(x)$ be its corresponding eigenfunction

Eigenfunctions ϕ satisfy $L\phi = \lambda\phi$ ⊛ where L is a linear operator. Let $\lambda = \mu + i\nu$

$$\Rightarrow L\phi = L(U+iV) = LU + iLV \overset{⊛}{=} \lambda\phi$$
$$= (\mu + i\nu)\phi$$
$$= \lambda U + i\lambda V$$
$$\Rightarrow LU + iLV = \lambda U + i\lambda V$$

equating Re, Im parts $\Rightarrow LU = \lambda U, \; LV = \lambda V \} \oplus$

b) 11.2.3 states: to each eigenvalue, corresponds to only one linearly indep. eigenfunction. So by $\oplus$ it follows that U, V must be linearly dependent. (have same eigenvalue λ)

c) By (b), if U, V are linearly dep.

$$\Rightarrow V = \alpha U \quad \text{for some const. } \alpha (\in \mathbb{C})$$
$$\Rightarrow \phi = U + iV$$
$$= U + i\alpha U$$
$$= U(1 + i\alpha) \quad \text{with } U \text{ a real valued function}$$

$\Rightarrow \phi$ is real apart from constant that may be complex (depending on value of α)

$\square$

Section 11.3

2) $y'' + 2y = -x \quad y(0)=0,\ y'(1)=0 \quad (-y'' = 2y + x)$ ⊛

By eqn 17 $\Rightarrow$ $y(x) = \dfrac{\sin\sqrt{2}\,x}{\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2}} - \dfrac{x}{2}$

By Ex 1) $y'' + \lambda y = 0 \quad y(0) = 0,\ y'(1) = 0$

$\Rightarrow \phi_n(x) = K_n \sin((n-\tfrac{1}{2})\pi x)$

$K_n = \sqrt{2}$

We assume $y = \sum_{n=1}^{\infty} b_n \phi_n(x)$ with $b_n = \dfrac{c_n}{\lambda_n - 2} = \dfrac{c_n}{(n-\frac{1}{2})^2\pi^2 - 2}$

c_n is the expansion coeff. from $f(x) = x$ (by ⊛)

$c_n = \dfrac{\sqrt{2}\,(-1)^{n+1}}{(n-\frac{1}{2})^2\pi^2}$ from 11.2 Prob 7

$\Rightarrow y = \sum_{n=1}^{\infty} b_n\phi_n(x) = \sum_{n=1}^{\infty} \dfrac{\sqrt{2}\,(-1)^{n+1}}{(n-\frac{1}{2})^2\pi^2} \cdot \dfrac{1}{(n-\frac{1}{2})^2\pi^2 - 2} \cdot \sqrt{2}\sin((n-\tfrac{1}{2})\pi x)$

$\Rightarrow y(x) = 2\sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}\sin((n-\frac{1}{2})\pi x)}{[(n-\frac{1}{2})^2\pi^2 - 2](n-\frac{1}{2})^2\pi^2}$

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3) $y'' + 2y = -x \quad y'(0) = 0,\ y'(1) = 0$

By Ex 3) $\phi_n(x) = K_n\cos(n\pi x) \quad K_n = \sqrt{2}$

We assume $y = \sum_{n=1}^{\infty} b_n\phi_n(x) \quad b_n = \dfrac{c_n}{\lambda_n - 2} = \dfrac{c_n}{n^2\pi^2 - 2}$

$c_n = \int_0^1 f(x)\phi_n(x)\,dx = \sqrt{2}\int_0^1 x\cos(\sqrt{\lambda_n}\,x)\,dx$

$= \sqrt{2}\,\dfrac{-1 + \cos(n\pi)}{(\sqrt{\lambda_n})^2} =$

$\Rightarrow y(x) = -\dfrac{1}{4} - \sum_{n=1}^{\infty} \dfrac{\cos((n-\frac{1}{2})\pi x)}{[(n-\frac{1}{2})^2\pi^2 - 2](n-\frac{1}{2})^2\pi^2}$

7) Determine the formal eigenfct series expansion

Assume f satisfies Thm 11.3.1

$y'' + \mu y = -f(x) \quad y'(0)=0, \ y(1)=0$

From prob 2, 11.2, for homog. problem

$\Rightarrow \phi_n(x) = \sqrt{2}\cos((n-\frac{1}{2})\pi x), \ \lambda_n = (n-\frac{1}{2})^2\pi^2$

From Thm 11.3.1, $y(x) = \sqrt{2}\sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu}\cos((n-\frac{1}{2})\pi x)$ (Eq 13 p 701)

for $\lambda_n \neq \mu$

$\Rightarrow c_n = \sqrt{2}\int_0^1 f(x)\cos((n-\frac{1}{2})\pi x)\,dx$ (Eq 9, p 701)

11) $y'' + 4\pi^2 y = a + x \quad y(0)=0, \ y(1)=0$

Determine for what value of a is there a soln.

Solving ODE using Undetermined coeffs.

$y(x) = \frac{a+x}{4\pi^2} + c_1\cos(2\pi x) + c_2\sin(2\pi x)$

$y(0) = 0 = \frac{a}{4\pi^2} + c_1 \Rightarrow c_1 = \frac{-a}{4\pi^2}$

$\Rightarrow y(x) = \frac{a+x}{4\pi^2} - \frac{a}{4\pi^2}\cos(2\pi x) + c_2\sin(2\pi x)$

$y(1) = 0 = \frac{a+1}{4\pi^2} - \frac{a}{4\pi^2} \Rightarrow \frac{1}{4\pi^2} = 0$ ⨳

$\Rightarrow$ no solution

⑤

19) $u_t = u_{xx} - x$ $\quad u(0,t)=0$, $u_x(1,t)=0$ $\quad u(x,0) = \sin\left(\frac{\pi x}{2}\right)$

B.C's $\Rightarrow$ $\phi_n(x) = \sqrt{2}\sin((n-\tfrac{1}{2})\pi x)$ $\quad \sqrt{\lambda_n} = (n-\tfrac{1}{2})\pi$

Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$ Sub into PDE, $\phi_n'' = -\lambda_n\phi_n(x)$

$$\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\phi_n''(x) - x = -\sum_{n=1}^{\infty}\lambda_n\phi_n(x)b_n(t) - x$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[b_n'(t) + \lambda_n b_n(t)\right]\phi_n(x) = -x \quad (*)$$

In Problem 2, $-x = \sum_{n=1}^{\infty} c_n\phi_n(x) = \sum_{n=1}^{\infty}\frac{\sqrt{2}(-1)^{n+1}}{(n-\frac{1}{2})^2\pi^2}\phi_n(x)$ $\quad = \alpha_n$

$$\Rightarrow (*) \text{ becomes } \sum_{n=1}^{\infty}\left[b_n'(t) + \lambda_n b_n(t) - \boxed{\frac{\sqrt{2}(-1)^{n+1}}{(n-\frac{1}{2})^2\pi^2}}\right]\phi_n(x) = 0$$

$\Rightarrow b_n'(t) + \lambda_n b_n(t) - \alpha_n = 0$ $\qquad$ $\boxed{\text{IC: } b_n(0) = 1 \ ?}$

$$\Rightarrow b_n(t) = \frac{\alpha_n}{\lambda_n} + Ce^{-\lambda_n t}$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x) = e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi x}{2}\right) - \sqrt{2}\sum_{n=1}^{\infty}\frac{\alpha_n}{(n-\frac{1}{2})^2\pi^2}\left[1-e^{-\sqrt{\lambda_n}t}\right]\cdot\sin((n-\tfrac{1}{2})\pi x)$$

$$\Rightarrow = \left[-\frac{\sqrt{2}\,4\alpha_1}{\pi^2} + \frac{\sqrt{2}e^{-\frac{\pi^2}{4}t}4\cancel{\alpha_1}}{\pi^2}\right]\sin\left(\frac{\pi x}{2}\right) + e^{-\frac{\pi^2}{4}t}\sin\left(\frac{\pi x}{2}\right) - \sqrt{2}\sum_{n=2}^{\infty}\frac{\alpha_n}{(n-\frac{1}{2})^2\pi^2}\left[1-e^{-\sqrt{\lambda_n}t}\right]\sin((n-\tfrac{1}{2})\pi x)$$

$$\Rightarrow = \sqrt{2}\left[-\frac{4\cancel{\alpha_1}}{\pi^2} + \frac{4\cancel{\alpha_1}}{\pi^2}e^{-\frac{\pi^2}{4}t} + \frac{1}{\sqrt{2}}e^{-\frac{\pi^2}{4}t}\right]\sin\left(\frac{\pi x}{2}\right)$$
$$\sqrt{2}\sum_{n=2}^{\infty}\frac{\alpha_n}{(n-\frac{1}{2})^2\pi^2}\left[1-e^{-\sqrt{\lambda_n}t}\right]\sin((n-\tfrac{1}{2})\pi x)$$

20) $u_t = u_{xx} + e^{-t}$ $\quad u_x(0,t) = 0$ $\quad u_x(1,t) + u(1,t) = 0$ $\quad u(x,0) = 1-x$

Homogeneous PDE $\Rightarrow u_t = u_{xx}$

Sep. of vars $\Rightarrow \phi''(x) + \lambda\phi(x) = 0$ $\quad \phi'(0) = 0$, $\phi'(1) + \phi(1) = 0$

From prob. 4, 11.2, we know $\phi_n(x) = \dfrac{\sqrt{2}}{\sqrt{1+\sin^2\sqrt{\lambda_n}}}\cos\sqrt{\lambda_n}\,x$

Where $\cos\sqrt{\lambda_n} - \sqrt{\lambda_n}\sin\sqrt{\lambda_n} = 0$. So similar to Ex. in Chapter 11.3,

Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$, by sub into PDE,

$$\sum_{n=1}^{\infty} b_n'(t)\phi_n(x) = \sum_{n=1}^{\infty} b_n(t)\phi_n''(x) + e^{-t} \overset{\phi_n'' = -\phi_n \cdot \text{Const}}{=\!=} -\sum_{n=1}^{\infty}\lambda_n b_n(t)\phi_n(t) + e^{-t}$$

$$\Rightarrow \sum_{n=1}^{\infty}\left[b_n'(t) - \lambda_n b_n(t)\right]\phi_n(x) = e^{-t} \;(*)$$

Note: $1 = \sum_{n=1}^{\infty} \underbrace{\boxed{\dfrac{\sqrt{2}\sin\sqrt{\lambda_n}}{\sqrt{\lambda_n}\sqrt{1+\sin^2\sqrt{\lambda_n}}}}}_{=\beta_n} \phi_n(x)$

$$\Rightarrow e^{-t} = \sum_{n=1}^{\infty}\beta_n e^{-t}\phi_n(x)$$

So $(*)$ ~~becomes~~ becomes $\sum_{n=1}^{\infty}\left[b_n'(t) + \lambda_n b_n(t) - \beta_n e^{-t}\right]\phi_n(x) = 0$

$$\Rightarrow \left[b_n'(t) + \lambda_n b_n(t) - \beta_n e^{-t}\right] = 0 \Rightarrow b_n' + \lambda_n b_n = \beta_n e^{-t} \;(+)$$

From Prob 12, $u(x,0) = 1-x = \sum_{n=1}^{\infty}\alpha_n\phi_n(x)$ $\quad \alpha_n = \dfrac{\sqrt{2}(1-\cos\sqrt{\lambda_n})}{\lambda_n\sqrt{1+\sin^2\sqrt{\lambda_n}}}$

$$\Rightarrow b_n(0) = \alpha_n$$

Solving 1st order ODE $(+)$, $b_n(t) = \dfrac{\beta_n(e^{-t} - e^{-\lambda_n t})}{\lambda_n - 1} + \alpha_n e^{-\lambda_n t}$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}\left[\frac{\beta_n(e^{-t} - e^{-\lambda_n t})}{\lambda_n - 1} + \alpha_n e^{-\lambda_n t}\right]\phi_n(x)$$

from ϕ_n $\qquad$ $= C_n$ in back of book

$$= \sqrt{2}\sum_{n=1}^{\infty}\left[\frac{\beta_n}{\lambda_n - 1}(e^{-t} - e^{-\lambda_n t}) + \alpha_n e^{-\lambda_n t}\right]\frac{\cos\sqrt{\lambda_n}\,x}{(1+\sin^2\sqrt{\lambda_n})^{1/2}}$$

$\blacksquare$

22) $u_t = u_{xx} + e^{-t}(1-x)$ $\quad u(0,t)=0$, $u_x(1,t)=0$ $\quad u(x,0)=0$ (6)

Following (Ex. 2) 11.3, Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\phi_n(x)$

$b_n' + \lambda_n b_n = \gamma_n(t)$ where

Prob 1 11.2 $\sqrt{\lambda_n} = (n-\frac{1}{2})\pi$

$$\gamma_n(t) = \int_0^1 e^{-t}(1-x)\,\phi_n(x)\,dx = e^{-t}\int_0^1 (1-x)\phi_n(x)\,dx$$

$$= e^{-t}\int_0^1 (1-x)\underbrace{\sqrt{2}\sin((n-\tfrac{1}{2})\pi x)}_{\phi_n(x)}\,dx = e^{-t}\underbrace{\sqrt{2}\,\frac{\sqrt{\lambda_n} - \sin\sqrt{\lambda_n}}{\lambda_n}}_{\alpha_n} = e^{-t}\alpha_n$$

$$\Rightarrow b_n' + \lambda_n b_n = \alpha_n e^{-t},\quad b_n(0)=0$$

$$\Rightarrow b_n(t) = \frac{\alpha_n e^{-t}}{\lambda_n - 1} + e^{-t\lambda_n}C \Rightarrow C = -\frac{\alpha}{\lambda_n - 1}$$

$$\Rightarrow b_n(t) = \frac{\alpha_n}{\lambda_n - 1}\left(e^{-t} - e^{-t\lambda_n}\right)$$

from ϕ_n

$$\Rightarrow u(x,t) = \sqrt{2}\sum_{n=1}^{\infty}\left[\frac{\alpha_n}{\lambda_n - 1}\left(e^{-t} - e^{-t\lambda_n}\right)\right]\sin((n-\tfrac{1}{2})\pi x)$$

$$\alpha_n = \sqrt{2}\,\frac{(n-\frac{1}{2})\pi - \sin((n-\frac{1}{2})\pi)}{(n-\frac{1}{2})^2\pi^2} \qquad \lambda_n = (n-\tfrac{1}{2})^2\pi^2$$

$$[p(x)y']' + q(x)y + \lambda r(x) y = 0$$

$y'' + \pi^2 y = 0 \Rightarrow \lambda > 0 \qquad y(0) = 0,\ y(1) = 0$

$\Rightarrow y(x) = c_1 \cos \pi x + c_2 \sin \pi x$

$BC \Rightarrow y(x) = c_2 \sin \pi x$

$y'' + \pi^2 y = a + x \qquad y_i(x) = (A_0 x + A_n)$

$y_i'(x) = A_0$

$y_i''(x) = 0$

$\Rightarrow 0 + \pi^2 (A_0 x + A_n) = a + x$

$\Rightarrow A_0 \pi^2 x + A_n \pi = x + a$

$\Rightarrow A_0 \pi^2 = 1 \Rightarrow A_0 = \frac{1}{\pi^2} \Rightarrow y_i = \frac{x}{\pi^2} + \frac{a}{\pi} \cancel{+ c_2 \sin t}$

$A_n \pi = a \Rightarrow A_n = \frac{a}{\pi}$

$y = y_i + y_H = \frac{x}{\pi^2} + \frac{a}{\pi} + c_2 \sin \pi x$

$0 = \frac{1}{\pi^2} + \frac{a}{\pi}$