HW 7 11.4) 1, 2, 3 ①
11.5) 2, 3, 5, 6, 7

11.4

1) $-(xy')' = \mu x y + f(x)$ y, y' bounded as $x \to 0$ $\boxed{y(1) = 0}$

f continuous on $0 \le x \le 1$ and μ is <u>not</u> an eigenvalue of the corresponding homog. problem.

ODE $\Rightarrow -xy'' - y' = \mu x y + f(x)$ or $-(xy')' = \mu x y + f(x)$ p 715 eqn (7)

$\Rightarrow$ Eigen function is $\phi_n(x) = J_0(\sqrt{\lambda_n}\, x)$ λ_n satisfies $J_0(\sqrt{\lambda}) = 0$

Let $\phi(x) = \sum_{n=0}^{\infty} b_n \phi_n(x)$, then

$$-(x\phi')' = \mu x \phi + f(x) = \mu x \phi + \frac{x f(x)}{x}$$

$$\Rightarrow -\left(x\left(\sum_{n=0}^{\infty} b_n \phi_n\right)'\right)' = \mu x \sum_{n=0}^{\infty} b_n \phi_n + \frac{x f(x)}{x}$$

$$-\left(x \sum_{n=0}^{\infty} b_n \phi_n'\right)' = -\sum_{n=0}^{\infty} b_n \phi_n' - x\sum_{n=0}^{\infty} b_n \phi_n'' = \mu x \sum_{n=0}^{\infty} b_n \phi_n + \frac{x f(x)}{x}$$

$\boxed{\text{Alternatively}}$: $\boxed{x^2 \phi'' + x\phi' + \mu x^2 \phi = -f(x)}$

zero order Bessel DE $\Rightarrow$ soln is $J_0(\sqrt{\lambda_n}\, x)$

$\frac{c_n}{\lambda_n - \mu}$, is the expansion coeff of $f(x)$ from previous HW use eqn in the section

$$y = \sum_{n=1}^{\infty} \frac{c_n}{\lambda_n - \mu} J_0(\sqrt{\lambda_n}\, x) \qquad c_n = \frac{\int_0^1 f(x) J_0(\sqrt{\lambda_n}\, x)\, dx}{\int_0^1 x J_0^2(\sqrt{\lambda_n}\, x)\, dx}$$

2) $-(xy')' = \lambda xy$ $\quad$ y, y' bdd as $x \to 0$

$y'(1) = 0$

3a) $(xy')' + y\left(\lambda x - \frac{k^2}{x}\right) = 0$ $\qquad \frac{dy}{dx} = \sqrt{\lambda}\frac{dy}{dt}, \quad \frac{d^2y}{dx^2} = \lambda \frac{d^2y}{dt^2}$

$t = \sqrt{\lambda}\, x \qquad x = \frac{t}{\sqrt{\lambda}}$

$$\frac{t}{\sqrt{\lambda}}\lambda \frac{d^2y}{dt^2} + \sqrt{\lambda}\frac{dy}{dt} + \lambda\frac{t}{\sqrt{\lambda}} y - \frac{k^2\sqrt{\lambda}}{t} y = 0$$

$$\Rightarrow t\sqrt{\lambda}\frac{d^2y}{dt^2} + \sqrt{\lambda}\frac{dy}{dt} + \sqrt{\lambda}\, t y - \frac{k^2\sqrt{\lambda}}{t} y = 0$$

$$\Rightarrow t\frac{d^2y}{dt^2} + \frac{dy}{dt} + ty - \frac{k^2}{t} y = 0$$

$$\Rightarrow t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} (t^2 - k^2) y = 0 \quad \text{Bessel's DE}$$

$\Rightarrow J_k(t)$ is solution (Y_k also soln, but unbounded)

b) $J_k(\sqrt{\lambda}\, x)$ satisfies the B.C. at $x = 0$

Other B.C. $\Rightarrow J_k(\sqrt{\lambda}) = 0$

So λ_n given by $\sqrt{\lambda_n}$ are positive zeros of $J_k(x)$

$$\Rightarrow \phi_n(x) = J_k(\sqrt{\lambda_n}\, x)$$

$\boxed{\lambda_n \phi_n(x) x = L[\phi_n]}$

c) If $L[y] = -(xy')' + \frac{k^2}{x} y \qquad r(x) = 1$

$$\lambda_n \int_0^1 x\phi_n(x)\phi_m(x)\,dx = \int_0^1 L[\phi_n]\phi_m\,dx$$
$$= \int_0^1 \phi_n(x) L[\phi_m]\,dx = \lambda_m \int_0^1 x\phi_n(x)\phi_m(x)\,dx$$

$$\Rightarrow (\lambda_n - \lambda_m)\int_0^1 x\phi_n\phi_m\,dx = 0 \Rightarrow \int_0^1 x\phi_n\phi_m\,dx = 0 \text{ for } m \neq n$$

d) Consider

$f(x) = \sum_{n=0}^{\infty} a_n\phi_n(x)$. Multiply by $x\phi_j(x)$

$$\Rightarrow \int_0^1 xf(x)\phi_j(x)\,dx = \int_0^1 \sum_{n=0}^{\infty} a_n x\phi_n(x)\phi_j(x)\,dx \overset{\text{part (c)}}{=\!=} a_j\int_0^1 x\phi_j(x)\phi_j(x)\,dx$$

$$\Rightarrow a_j = \frac{\int_0^1 x f(x)\phi_j(x)\,dx}{\int_0^1 x\phi_j^2(x)\,dx} \qquad j = 1, 2, \ldots$$

$$-(xy')' + \frac{k^2}{x} y = \mu x y + f(x) \qquad y, y' \text{ bdd as } x \to 0$$
$$y(1) = 0$$

$$\text{et } \phi(x) = \sum_{n=0}^{\infty} b_n \phi_n(x) \qquad \phi_n(x) = J_k(\sqrt{\lambda_n}\, x) \text{ from part (a)}$$

$$L[\phi] = \mu x \phi + f(x) = \mu x \phi + x \frac{f(x)}{x}$$

11.5

2) $r=1$, $u(1,t)=0$ $t\geq 0$ I.C.) $u(r,0)=0$ $0\leq r\leq 1$

$u_t(r,0)=g(r)$ $g(1)=0$

Follow example in chapter 11.5 p 722–724

only difference is $\cancel{\cos}$ sin $\lambda_n a t$ instead of $\cos \lambda_n a t$

3) $u(r,t)$ is a vib. circ. elastic membrane $r=1$

$u(1,t)=0 \qquad u(r,0)=f(r) \qquad f(1)=g(1)=0$

$u_t(r,0)=g(r)$

p. 722, Eqns (2) through (17) is the derivation of solution

$a^2\left(u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}\right)=u_{tt}$

$$\boxed{\begin{aligned}&\Rightarrow \phi_n(x)=J_0(\lambda_n r)\\&\Rightarrow u(r,t)=\sum_{n=1}^{\infty}\left[c_n J_0(\lambda_n r)\cos(\lambda_n a t)+k_n J_0(\lambda_n r)\sin(\lambda_n a t)\right]\end{aligned}}$$

$IC \Rightarrow u(r,0)=\sum_{n=1}^{\infty} c_n J_0(\lambda_n r)=f(r)$

$u_t(r,0)=\sum_{n=1}^{\infty} a\lambda_n k_n J_0(\lambda_n r)=g(r)$

Multiply by $\cancel{f}\, J_0(\lambda_n r)$ and integrate $\int_0^1$

$$\Rightarrow \boxed{\begin{aligned}c_n&=\frac{\int_0^1 \cancel{r} f(r) J_0(\lambda_n r)\,dr}{\int_0^1 r J_0^2(\lambda_n r)\,dr}\\ k_n&=\frac{\int_0^1 r g(r) J_0(\lambda_n r)\,dr}{\int_0^1 r\,\cancel{(\ )}\, J_0^2(\lambda_n r)\,dr\ a\lambda_n}\end{aligned}} \qquad n=1,2,\ldots$$

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5) a) $x=r\cos\theta$
$y=r\sin\theta$
$z=z$

$u_{rr}+(1/r)u_r+\left(\frac{1}{r^2}\right)u_{\theta\theta}+u_{zz}=0$

Just ~~&~~ use sep. of variables $u(r,\theta,z)=R(r)\Theta(\theta)Z(z)$

b) No Θ-dependence $\Rightarrow u_{\theta\theta}=0 \Rightarrow u_{rr}+\left(\frac{1}{r}\right)u_r+u_{zz}=0$

Solve using $u(r,z)=R(r)Z(z)$

$\Rightarrow r^2R''+rR'+\lambda^2 r^2 R=0$

$Z''-\lambda^2 Z=0$

6) $0 < z < \infty$ θ independent
$0 \le r < 1$ $z \to \infty$

Assume $u(r,z)$ satisfies $u(1,z)=0$, $u(r,0)=f(r)$ $u(r,z) = R(r)Z(z)$

Using #5,

$$r^2R'' + rR' + \lambda^2 r^2 R = 0$$

Bessel eqn with $\nu = 0$

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

Change of vars $z = \sqrt{\lambda}\, r$ (?)
something like this

Same as #3, 11.4 $k=0$

$\Rightarrow J_k(\lambda_n r)$ is the solution

$k=0 \Rightarrow c_2 J_0(\lambda_n r) = R(r)$

$$Z'' - \lambda^2 Z = 0$$

$$\Rightarrow \alpha^2 - \lambda^2 = 0$$

$$\Rightarrow \alpha^2 = \lambda^2$$

$$\Rightarrow \alpha = \pm\lambda$$

$$\Rightarrow Z(z) = c_1 e^{-\lambda_n z} + c_2 e^{\lambda_n z}$$

as $z \to \infty$ soln decays

$$\Rightarrow Z(z) = c_1 e^{-\lambda_n z}$$

$\star$ λ_n come from

$$\Rightarrow u(r,z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} J_0(\lambda_n r)$$

c_n is found by

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} c_n J_0(\lambda_n r)$$

Multiply by $J_0(\lambda_n r)x$ and integrate from 0 to 1

$$u(r,z) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n z} J_0(\lambda_n r) \qquad c_n = \frac{\int_0^1 r J_0(\lambda_n r) g(r)\, dr}{\int_0^1 r J_0^2(\lambda_n r)\, dr}$$

7) $V_{xx} + V_{yy} + k^2 V = 0$

a) $V_{rr} + \left(\frac{1}{r}\right)V_r + \left(\frac{1}{r^2}\right)V_{\theta\theta} + k^2 V = 0$

$V(r,\theta) = R(r)\,\Theta(\theta)$

$\Rightarrow$ Separation of Vars.

$r^2R'' + rR' + (k^2r^2 - \lambda^2)R = 0$ (A) $\qquad \Theta'' + \lambda^2\Theta = 0$ (B)

b) $r < c$ disk, periodic period 2π

$V(c,\theta) = f(\theta) \quad 0 \le \theta < 2\pi$

(B) $\Rightarrow \Theta(\theta) = c_1 \cos(\lambda_m \theta) + c_2 \sin(\lambda_m \theta)$ $\quad \sqrt{\lambda} = m \Rightarrow \lambda = n^2$

(Eigenvalue Problem) $\lambda = m$ (think for radial problem before)

(A) $\Rightarrow r^2R'' + rR' + ((kr)^2 - \lambda^2)R = 0$

$\Rightarrow$ Bessel's equation order $\lambda = m$

$\Rightarrow$ Solution $R(r) = \tilde{C}\, J_m(kr)$

$\Rightarrow$ product soln is $V(r,\theta) = \sum_{m=0}^{\infty} J_m(kr)\left(b_m \sin(m\theta) + c_m \cos(m\theta)\right)$

$\Rightarrow J_m(kc)\, b_m = \frac{2}{2\pi}\int_0^{2\pi} f(\theta)\sin(m\theta)\, d\theta$

$J_m(kc)\, c_m = \frac{2}{2\pi}\int_0^{2\pi} f(\theta)\cos(m\theta)\, d\theta$

$\Downarrow$

$\frac{a_0}{2}$

$V(r,\theta) = \frac{1}{2} c_0 J_0(kr) + \sum_{m=1}^{\infty} J_m(kr)\left(b_m \sin(m\theta) + c_m \cos(m\theta)\right)$

$b_m = \frac{1}{\pi J_m(kc)}\int_0^{2\pi} f(\theta)\sin(m\theta)\, d\theta$

$c_m = \frac{1}{\pi J_m(kc)}\int_0^{2\pi} f(\theta)\cos(m\theta)\, d\theta$