MATH 146C - Review of Chapters 10 and 11

Chapter 10

Eigenvalue Problem

$y'' + \lambda y = 0$ (Note: this is with "+" sign)

$\lambda > 0 \Rightarrow \pm i\sqrt{\lambda} \Rightarrow y(x) = c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x)$

$\lambda = 0 \Rightarrow 0, 0 \Rightarrow y(x) = c_1 + c_2 x$

$\lambda < 0 \Rightarrow r = \pm\sqrt{-\lambda} \Rightarrow y(x) = c_1 \cosh(\sqrt{\lambda} x) + c_2 \sinh(\sqrt{\lambda} x)$

Remark: remember for $(y'' - \lambda y = 0)$ $\lambda < 0$ and $\lambda > 0$ are reversed

Given

$\phi''(x) + \lambda \phi(x) = 0$, boundary conditions yield

B.C.'s	$\phi(0) = 0$, $\phi(L) = 0$	$\phi_x(0) = 0$, $\phi_x(L) = 0$	$\phi(0) = 0$, $\phi_x(L) = 0$	$\phi_x(0) = 0$, $\phi(L) = 0$
Eigenvalues	$\left(\frac{n\pi}{L}\right)^2$	$\left(\frac{n\pi}{L}\right)^2$	$\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2$	$\left(\frac{(n-\frac{1}{2})\pi}{L}\right)^2$
Eig. fncts	$\sin\left(\frac{n\pi x}{L}\right)$	$\cos\left(\frac{n\pi x}{L}\right)$	$\sin\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$	$\cos\left(\frac{(n-\frac{1}{2})\pi x}{L}\right)$

Orthogonality Conditions

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0 \quad \forall m, n$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

Fourier Series Formula

$$f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos\left(\frac{m\pi x}{L}\right) + b_m \sin\left(\frac{m\pi x}{L}\right)\right)$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

* Know how to sketch Fourier series

Fourier Sine and Cosine Series

Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Sine Series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

* Know how to sketch Fourier sine/cosine series

Separation of Variables

For Heat equation let $u(x,t) = \phi(x)h(t)$

Wave " " $u(x,t) = \phi(x)h(t)$

Laplace " " $u(x,y) = \phi(x)\psi(y)$

(*)

Heat equation $u_t = u_{xx}$

Wave equation $u_{tt} = u_{xx}$

Laplace equation $u_{xx} + u_{yy} = 0$ or $\nabla^2 u = 0$

To do separation of variables, just ~~set~~ substitute (*) into the PDE and separate the functions

Heat Equation

Transient / Steady state

Steady state: $v(x) = (T_2 - T_1)\frac{x}{L} + T_1$

$u(x,t) = v(x) + w(x,t)$

$w(x,t)$ satisfies the PDE $\alpha^2 w_{xx} = w_t$

Remember to solve for BC.! $\quad w(0,t) = u(0,t) - v(0)$

$w(L,t) = u(L,t) - v(L)$

IC: $\quad w(x,0) = u(x,0) - v(x)$

Wave Equation

D'Alembert's Solution

$$u(x,t) = \frac{1}{2}\left[f(x-at) + f(x+at)\right] + \frac{1}{2a}\int_{x-at}^{x+at} g(\xi)\,d\xi$$

Laplace Equation

Note: $y(x) = c_1 \sinh\left(\frac{n\pi}{L}(b-y)\right) + c_2 \cosh\left(\frac{n\pi}{L}(b-y)\right)$

also solves the ODE, so transltion works here

when you have $y(0) = h(x)$
$y(b) = 0$
as boundary conditions

(see 10.8 #2 in Homework)

Equations in Polar Coordinates

Laplace Equation $\Rightarrow$ $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

$\Rightarrow$ For sep. of vars $u(r,\theta) = R(r)\,\Theta(\theta)$

$\Rightarrow$ $\underbrace{r^2 R'' + r R' - \lambda R = 0}_{A}$ and $\Theta'' + \lambda \Theta = 0$

A is an Euler DE with solution

$$R(r) = k_1 r^{\sqrt{\lambda}} + k_2 r^{-\sqrt{\lambda}} \qquad (\lambda > 0)$$

Note: Throw out $k_1 r^{\sqrt{\lambda}}$ when region is outside circle
Throw out $k_2 r^{-\sqrt{\lambda}}$ when inside circle

Chapter 11

Sturm-Liouville Form

$[p(x)y']' - q(x)y + \lambda r(x) y = 0$ (*)

Notation (Linear operator form)

$L[y] = -[p(x)y']' + q(x)y$

So $L[y] = \lambda r(x) y$

Orthogonality Condition

$\int_0^1 r(x)\,\phi_m(x)\,\phi_n(x)\,dx = 0$ for $m \neq n$

Normalization Condition

$\int_0^1 r(x)\,\phi_n^2(x)\,dx = 1$

$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ is the expansion of $f(x)$ as a series.

with $c_n = \int_0^1 f(x)\,\phi_n(x)\,dx$ (+)

Non-Homogeneous BVP $y'' + \mu y = f(x)$

* Solve homogeneous equation with B.C.'s
* Soln is of the form $y = \sum_{n=1}^{\infty} b_n \phi_n(x)$

where $b_n = \dfrac{c_n}{\lambda_n - \mu}$ ϕ_n is eigen functions, λ_n is eigenvalues

c_n are coefficients from expansion of $f(x)$ found in (+)

Non-Homogeneous Heat Equation

$u_t = u_{xx} + f(x,t)$ with given B.C./I.C.

① Find eigenfunction by solving the homogeneous problem (Sep. of Vars, etc.) $\phi_n(x)$. Normalize using formula.

② Assume $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$

③ Find $b_n(t)$. Multiple methods to do this (see 11.3 #19, 20, 22)
All involve solving a 1^{st} order ODE

④ Write solution $u(x,t) = \sum_{n=1}^{\infty} b_n(t)\,\phi_n(x)$

Bessel D.E. Stuff