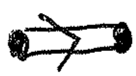
Weyl Groups

The Lie groups

 $D_2 = \mathrm{Spin}(4)$

&

 $B_2 = \mathrm{Spin}(5)$

have isomorphic (as Lie groups) maximal tori T, & therefore have isom. lattices:

$$L = \{ x \in \mathfrak{t} : \exp(2\pi x) = 1 \in T \}$$

But they have nonisomorphic Weyl groups:

$$W = N(T)/T$$

Let's work these out & maybe see

$$W(B_n) \cong S_n \ltimes \mathbb{Z}_2^n$$

$$W(D_n) \cong S_n \ltimes \mathbb{Z}_2^{(n-1)}$$

where the action of S_n on $\mathbb{Z}_2^n$ is obvious, but that of S_n on $\mathbb{Z}_2^{(n-1)}$ is not!

Let's look at $D_2 = \mathrm{Spin}(4)$ & then $B_2 = \mathrm{Spin}(5)$.

For $SO(4)$

$$T = \left\{ \left(\begin{array}{c|c} \exp(\theta_1 J) & 0 \\ \hline 0 & \exp(\theta_2 J) \end{array} \right) : \theta_1, \theta_2 \in \mathbb{R} \right\}$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

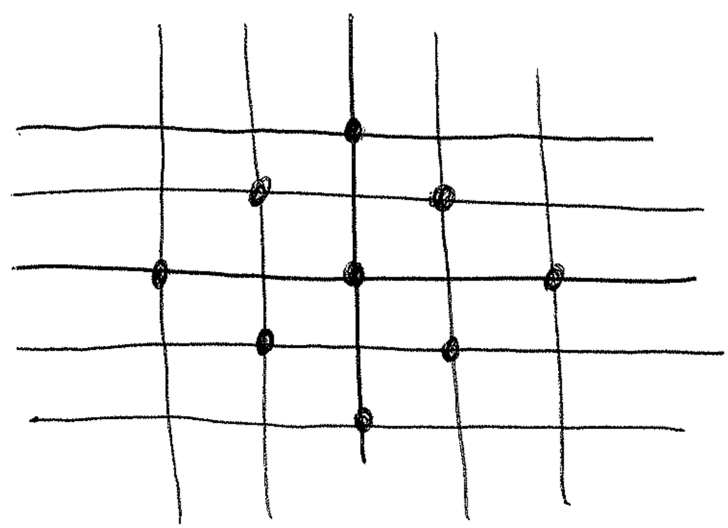
$$\exp(\theta J) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathfrak{t} = \left\{ \begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix} : \theta_1, \theta_2 \in \mathbb{R} \right\}$$

$$L = \left\{ \begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix} : \theta_1, \theta_2 \in \mathbb{Z} \right\}$$

For $D_2 = \mathrm{Spin}(4)$, $\mathfrak{t}$ is the same, but

$$L = \left\{ \begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix} : \theta_1 + \theta_2 \in 2\mathbb{Z},\ \theta_i \in \mathbb{Z} \right\}$$

$L =$ 

So $W(D_2)$ is at most the dihedral group of the square, which has 8 elts

$$g \exp(tx) g^{-1} = \exp(t\, g x g^{-1}) \qquad \begin{array}{l} x \in so(n) \\ g \in SO(n) \end{array}$$

winds up implying $g \in N(T)$ iff $gxg^{-1} \in \mathfrak{t}$ $\forall x \in \mathfrak{t}$.

In fact $N(T)/T$ is the same for $SO(n)$ & its double cover $Spin(n)$.

So we can find elts of $W(D_2)$ by finding $g \in SO(4)$ s.t. $x \in \mathfrak{t} \Rightarrow gxg^{-1} \in \mathfrak{t}$

i.e., find $g \in SO(4)$ s.t.

$$g \left(\begin{array}{c|c} \theta_1 J & 0 \\ \hline 0 & \theta_2 J \end{array} \right) g^{-1} = \left(\begin{array}{c|c} ? & 0 \\ \hline 0 & ? \end{array} \right)$$

For example

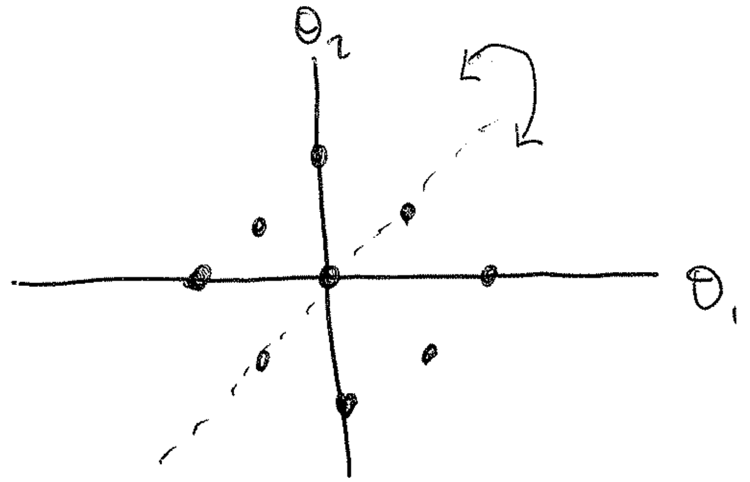
$$g = \left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\multirow{2}{*}{0}} & 1 & 0 \\ & & 0 & 1 \\ \hline 1 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ 0 & 1 & & \end{array}\right) \in SO(4)$$

$$g = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$g^{-1} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} \theta_2 J & 0 \\ 0 & \theta_1 J \end{pmatrix}$$

So $g \in N(T)$ & $[g] \in W = N(T)/T$ is nontrivial.

$L =$ 

This $[g]$ is reflection across the $\theta_1 = \theta_2$ line, since this does $(\theta_1, \theta_2) \longmapsto (\theta_2, \theta_1)$.

Anyone of good taste should want

$$(\theta_1, \theta_2) \longmapsto (-\theta_2, -\theta_1)$$

the reflection across the $\theta_1 = -\theta_2$ line, to also be in W.

Want $g \in SO(4)$,

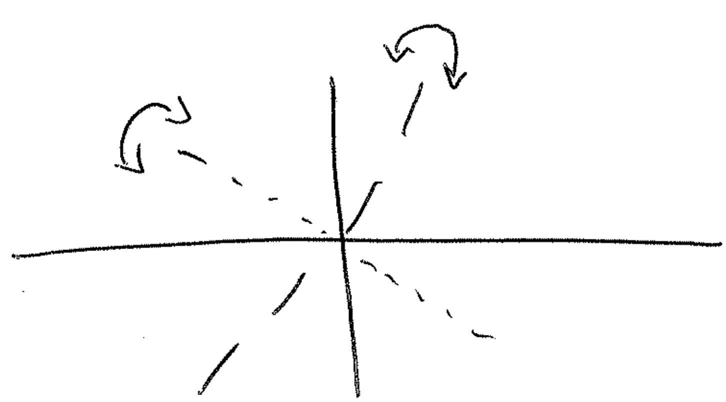
$$g \begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix} g^{-1} = \begin{pmatrix} -\theta_2 J & 0 \\ 0 & -\theta_1 J \end{pmatrix}$$

Let $K = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, & take $g = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$.

Then $KJK^{-1} = -J$, &

$$\begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} \theta_1 J & 0 \\ 0 & \theta_2 J \end{pmatrix} \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} = \begin{pmatrix} -\theta_2 J & 0 \\ 0 & -\theta_1 J \end{pmatrix}$$

Thus $W(D_2)$ contains 2 reflections:



& in fact is generated by these, so it's $\mathbb{Z}_2 \times \mathbb{Z}_{2p}$

Now try $B_2 = \mathrm{Spin}(5)$. In $SO(5)$

$$T = \left\{ \left(\begin{array}{c|c|c} \exp(\theta_1 J) & 0 & 0 \\ \hline 0 & \exp(\theta_2 J) & 0 \\ \hline 0 & 0 & 1 \end{array} \right) : \theta_1, \theta_2 \in \mathbb{R} \right\}$$

$\cong T$ for $SO(4)$. So we get the same $L \subseteq \mathfrak{t}$ as well. But W is bigger now.

In $SO(3)$ we have a rotation that acts as a reflection on $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$

$(x,y) \longmapsto (x,y,z)$

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -1 \end{array} \right) = \left(\begin{array}{c|c} K & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0\ 0 & -1 \end{array} \right)$$

Use this idea to get an elt of $SO(5)$ that's not in $SO(4) \hookrightarrow SO(5)$

$$X \mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix}$$

Like

$$g = \begin{pmatrix} I & & 0 \\ & K & \\ 0 & & -1 \end{pmatrix} = g^{-1}$$

Then

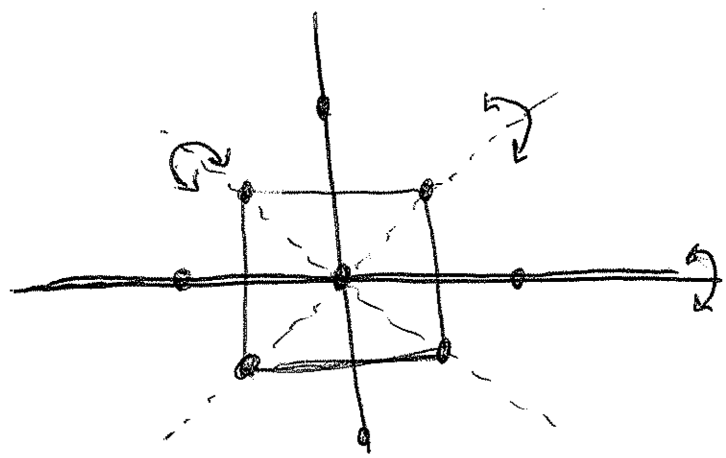
$$g \begin{pmatrix} \theta_1 J & & 0 \\ 0 & \theta_2 J & \\ & & 0 \end{pmatrix} g^{-1} = \begin{pmatrix} \theta_1 IJI & & 0 \\ 0 & \theta_2 KJK & \\ & & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \theta_1 J & 0 & 0 \\ 0 & -\theta_2 J & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So $[g]$ gives the reflection

$$(\theta_1, \theta_2) \longmapsto (\theta_1, -\theta_2)$$

& the group generated by the three reflections:



is the full dihedral group of the square;

this is W for $B_2 = \mathrm{Spin}(5)$.