Weyl Group Orbits

Thm If G is a compact semisimple Lie group & T is a maximal torus, every $g \in G$ is conjugate to some $t \in T$. Also if $t, t' \in T$ are conjugate in G then

$$t' = g t g^{-1}$$

for some $g \in N(T)$.

Since G acts on itself by conjugation, it acts on $\mathfrak{g}$:

$$\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g})$$

Every $x \in \mathfrak{g}$ is "conjugate" to some $t \in \mathfrak{t}$:

$$x = \mathrm{Ad}(g)t \quad \text{for some } g \in G.$$

Also if t & $t' \in \mathfrak{t}$ are "conjugate" in $\mathfrak{g}$, i.e. $t' = \mathrm{Ad}(g)t$ for some $g \in G$, then we can actually choose $g \in N(T)$.

Recall $N(T)$ acts on T by conjugation & thus on $\mathfrak{t}$:

$$\mathrm{Ad}\big|_{N(T)} : N(T) \longrightarrow \mathrm{Aut}(\mathfrak{t})$$

Since $T \subseteq N(T)$ acts trivially on itself by conjugation, the Weyl group

$$W = N(T)/T$$

acts on T by conjugation & on $\mathfrak{t}$ by Ad.

So by our thm:

$$\{\text{conjugacy classes in } G\} \cong \{W \text{ orbits in } T\}$$

& also

$$\{\text{adjoint orbits in } \mathfrak{g}\} \cong \{W \text{ orbits in } \mathfrak{t}\}$$

Examples

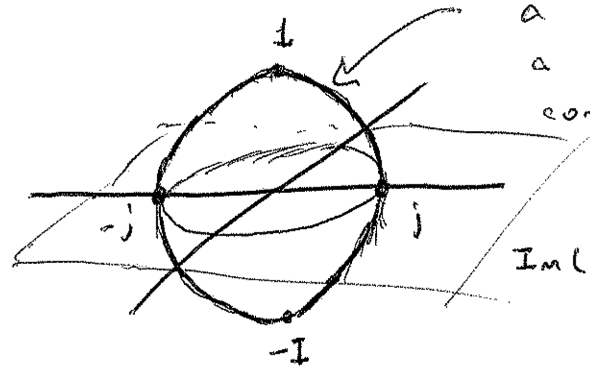
$A_1 \cong \mathfrak{su}(2)$. Here $G = SU(2)$

$T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in \mathbb{R} \right\}$

$SU(2) \cong \{\text{unit quaternions}\} \cong S^3 \subseteq \mathbb{H}$

a max. torus is a great circle containing $1 \in SU(2)$;

SU(2)

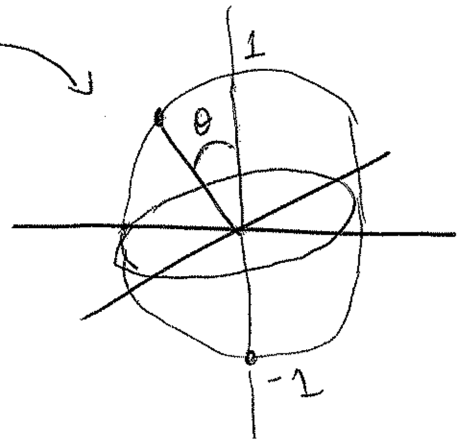


$\mathrm{Im}(\mathbb{H}) = \{\alpha i + \beta j + \gamma k\}$

Each unit imaginary quaternion gives such a circle; $\mathbb{RP}^2$ is the space of maximal tori.

$$Spin(3) \cong SU(2) \xrightarrow[\text{\& onto}]{2-1} SO(3)$$

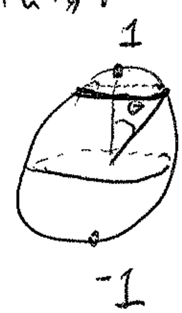
Maps to rotation by 2θ.



$Z(SU(2)) = \{\pm 1\}$, $\qquad SU(2)/\{\pm 1\} \cong SO(3)$

Two elements of $SO(3)$ are conjugate iff they're rotations about some axis by the same (unsigned) angle.

Conjugacy classes in $SU(2)$ are (typically) 2-spheres like this:

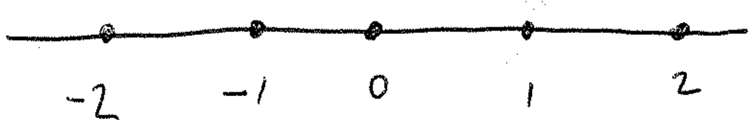


Also: $\{1\}$ & $\{-1\}$ are conjugacy classes.

So

$$\{\text{conjugacy classes in } SU(2)\} \cong [0, \pi]$$

Another way to see it:



$$T \cong S^1$$

$$\mathfrak{t} \cong \mathbb{R}$$

$$\cup\!\!|$$

$$L$$

$$L \subseteq \mathfrak{t} \qquad \mathfrak{t}/L \cong T$$

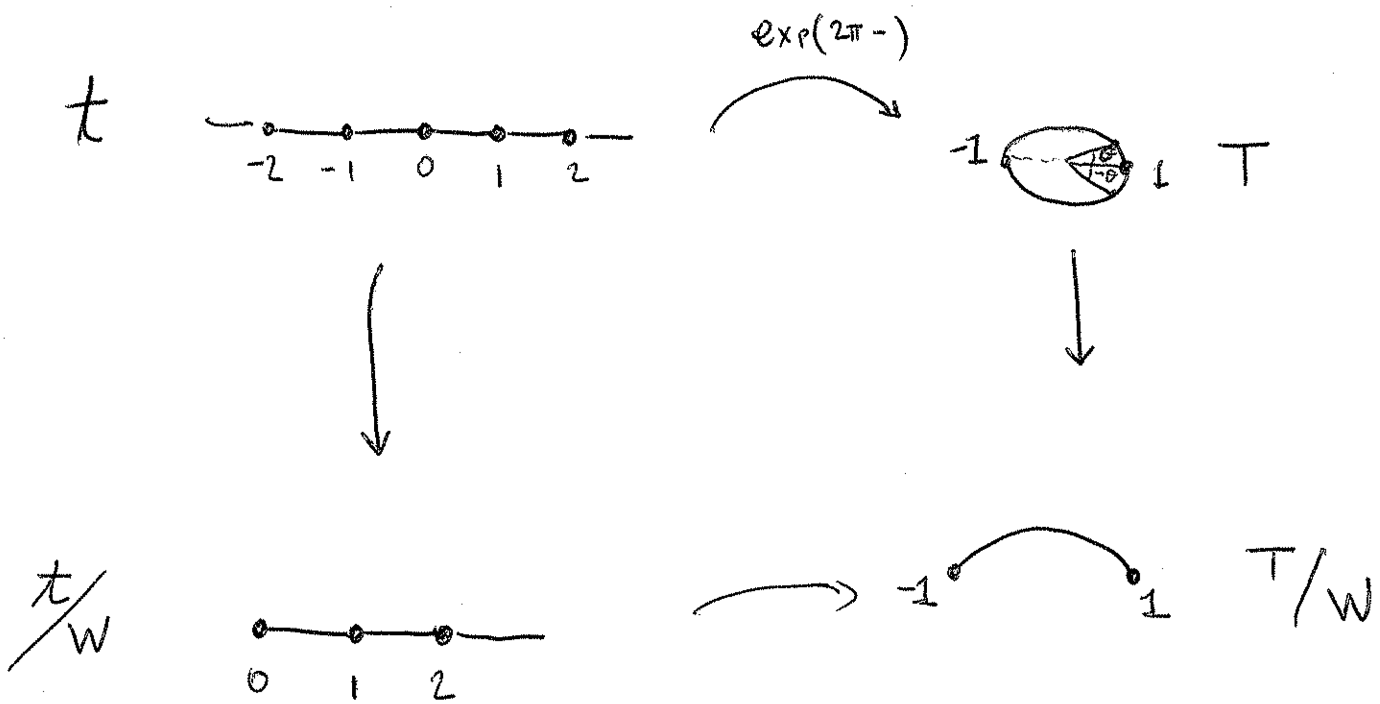
For A_n, the Weyl group is S_{n+1}.

So for A_1, it's $S_2 \cong \mathbb{Z}_2$, acting as reflection on $\mathfrak{t}$.

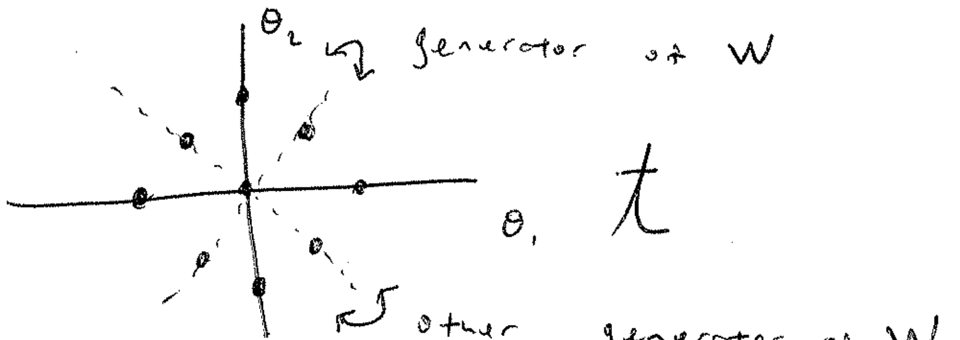
So:

$\{\text{adjoint orbits in } \mathfrak{g}\} \cong \{W \text{ orbits in } \mathfrak{t}\} \cong [0,\infty)$

$\{\text{conjugacy classes in } G\} \cong \{W \text{ orbits in } T\} \cong \{W \text{ orbits in } \mathfrak{t}/L\} \cong [0,\pi]$



- D_2 $\quad SO(4)$ $\quad Spin(4)$



$W \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

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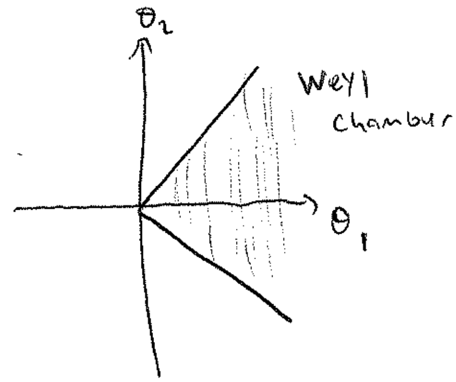
What's

$\{$adjoint orbits in $\mathfrak{g}\} \cong \{W$ orbits in $\mathfrak{t}\} =$ 

$= \{(\theta_1, \theta_2) \in \mathbb{R}^2 : 0 \leq \theta_1 \leq |\theta_2|\}$

A <u>Weyl chamber</u> is a fundamental region for the W action on $\mathfrak{t}$ whose "walls" are the "mirrors" for the reflections generating W.

What's

$\{$conj. classes in $G\} = \{W$ orbits in $T\} =$